CONTENTS

Physics. — *Critical speeds of rotating shafts.* By C. B. Biezeno.

1. *Introduction.* It is a well-known fact, that a rotating shaft, supported either in a statically determinate or a statically undeterminate way and bearing n centred and concentrated masses m_1, $m_2 \ldots m_n$ (in comparison of which the mass of the shaft itself be neglected), may whirl at most at n different angular speeds ω. However, up to now, no satisfactory proof has been given for the fact — tacitly assumed by all technicians —, that n, just now denoted as the upper limit of the number of unequal whirling speeds, in reality represents this number exactly. As will be seen in par. 2 the reciprocal squares of the critical angular speeds are roots of a so-called secular-equation of the nth degree and therefore it only needs evidence that all these roots are *real, positive* and *unequal.* This communication deals with the unequality of the roots; only for completeness' sake the proof of their reality and positiveness, which can be found elsewhere, will be included [1].

2. *The secular-equation having for roots the reciprocal squares of the critical angular speeds.* If at a whirling speed ω of the shaft the masses m_i $(i=1, 2 \ldots n)$ have deflections y_i $(i=1, 2 \ldots n)$, then it is evident that these deflections are maintained by the mass-reactions $m_i \omega^2 y_i$ $(i=1, 2, \ldots n)$. Therefore the following equations — where a_{ij} represents the ordinary Maxwell-number with respect to the points i and j — hold:

$$\sum_{j=1}^{n} m_j \omega^2 y_j a_{ij} = y_i \qquad (i=1, 2 \ldots n). \qquad (1)$$

They only admit a solution y_i $(i=1, 2 \ldots n)$ different from zero if

$$\begin{vmatrix} m_1 \omega^2 a_{11}-1 & m_2 \omega^2 a_{12} & m_3 \omega^2 a_{13} \ldots m_n \omega^2 a_{1n} \\ m_1 \omega^2 a_{21} & m_2 \omega^2 a_{22} & m_3 \omega^2 a_{23} \ldots m_n \omega^2 a_{2n} \\ \vdots & & \\ m_1 \omega^2 a_{n1} & m_2 \omega^2 a_{n2} & m_3 \omega^2 a_{n3} \ldots m_n \omega^2 a_{nn} - 1 \end{vmatrix} = 0. \quad (2)$$

[1] See f.i. C. B. Biezeno and R. Grammel, Technische Dynamik, p. 807 (J. Springer, Berlin).

If we put $\dfrac{1}{\omega^2}=z$ and $\sqrt{m_i m_j a_{ij}}=a_{ij}$, this equation is equivalent with:

$$S_n \equiv \begin{vmatrix} a_{11}-z & a_{12} & a_{13}\ldots a_{1n} \\ a_{21} & a_{22}-z & a_{23}\ldots a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} & a_{n3}\ldots a_{nn}-z \end{vmatrix} = 0 \quad \ldots \quad (3)$$

the determinant S_n being symmetrical with respect to its main diagonal as result of the MAXWELL-relation $a_{ij}=a_{ji}$. To prove that equation (3) has no *equal* roots, it will do to show that the two equations

$$T \equiv S_{n-1} \equiv \begin{vmatrix} a_{11}-z & a_{12} & \ldots a_{1,\,n-1} \\ a_{21} & a_{22}-z & \ldots a_{2,\,n-1} \\ \vdots & & \\ a_{n-1,\,1} & a_{n-1,\,2} & \ldots a_{n-1,\,n-1}-z \end{vmatrix} = 0 \quad \ldots \quad (4a)$$

and

$$U \equiv \begin{vmatrix} a_{12} & a_{13}\ldots \ldots \ldots \ldots \ldots a_{1n} \\ a_{22}-z & a_{23}\ldots \ldots \ldots \ldots \ldots a_{2n} \\ \vdots & \\ a_{n-1,\,2} & a_{n-1,\,3}\ldots \;a_{n-1,\,n-1}-z \quad a_{n-1,\,n} \end{vmatrix} = 0 \quad \ldots \quad (4b)$$

have no *common* roots. For, after a well-known theorem, a double root of equation (3) is a common root of all equations, which may be derived from (3) by putting its minors of the $(n-1)^{\text{th}}$ degree equal to zero. Therefore if only two of these equations, such as (4a) and (4b) are shown to have no common roots, it is impossible that (3) should have a double root. The non-existence of common roots of (4a) and (4b) will be proved in par. 5 by demonstrating that all roots of (4a) are real and positive and that all real roots of (4b) are negative.

3. *Some properties of the* MAXWELL *numbers* a_{ij}. We design by $\triangle_n$ the determinant of the nth order

$$\triangle_n \equiv \begin{vmatrix} a_{11} & a_{12}\ldots a_{1n} \\ a_{21} & a_{22}\ldots a_{2n} \\ \vdots & \\ a_{n1} & a_{n2}\ldots a_{nn} \end{vmatrix}, \quad \ldots \ldots \ldots \quad (1)$$

by $\triangle_n^{p,q,r....}$ (with $i,j,k....$ above) the determinant composed of those elements of $\triangle_n$, which are found in the rows $i,j,k...$ and the columns p,q,r; more particular by $\triangle_n^{2,3...n}$ (with $1,2...n-1$ above) the determinant derived from $\triangle_n$ by suppressing the nth row and the first column. Furthermore the symbol $a_{ij}^{u,v,w...}$ will stand for the MAXWELL number a_{ij}, relating to the points (i,j) that appears if, in addition to the natural supports of the shaft, artificial ones are introduced in the points $u,v,w....$

Now we consider the shaft under the action of a unit-force $P=1$ at the point n and ask for the displacement $a_{n,n}^{1,2...(n-1)}$ that occurs if in all points $1,2...(n-1)$ artificial supports are introduced. Let $X_1, X_2...X_{n-1}$ be the unknown reactions in those points. Then we have

$$\left.\begin{aligned}
y_1 &= X_1\,a_{11} + X_2\,a_{12} + ... + X_{n-1}\,a_{1,n-1} + a_{1,n} = 0 \\
y_2 &= X_1\,a_{21} + X_2\,a_{22} + ... + X_{n-1}\,a_{2,n-1} + a_{2,n} = 0 \\
y_{n-1} &= X_1\,a_{n-1} + X_2\,a_{n-1,2} + ... + X_{n-1}\,a_{n-1,n-1} + a_{n-1,n} = 0 \\
y_n &= X_1\,a_{n,1} + X_2\,a_{n,2} \quad + ... + X_{n-1}\,a_{n,n-1} + a_{n,n} = a_{n,n}^{1,2...(n-1)}
\end{aligned}\right\} \quad (2)$$

and it follows at once — $X_1, X_2...X_{n-1}$ and $a_{n,n}^{1,2...(n-1)}$ being regarded as unknowns —

$$a_{n,n}^{1,2...(n-1)} = \frac{\triangle_n}{\triangle_{n-1}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We learn from (3) that $\triangle_n$ and $\triangle_{n-1}$ have similar signs because $a_{n,n}^{1,2...(n-1)}$ is essentially positive, and from this fact again it follows, that all determinants $\triangle_n, \triangle_{n-1},...\triangle_1$ have the same sign and that they all must be positive because the last one, $\triangle_1 \equiv a_{11}$, is essentially positive.

Once more we consider the shaft under the action of a unit force P at the point n but now ask for the deflection $a_{1,n}^{2,3...k}$ in the point 1, which occurs if in the points $2,3...k$ artificial supports are introduced. If these supports exert the reaction $X_2, X_3...X_k$, we have

$$\left.\begin{aligned}
y_1 &= X_2\,a_{12} + X_3\,a_{13} + ... + X_k\,a_{1k} + a_{1,n} = a_{1,n}^{2,3...k} \\
y_2 &= X_2\,a_{22} + X_3\,a_{23} + ... + X_k\,a_{2k} + a_{2,n} = 0 \\
y_3 &= X_2\,a_{32} + X_3\,a_{33} + ... + X_k\,a_{3k} + a_{3,n} = 0 \\
&\;\vdots \\
y_k &= X_2\,a_{k2} + X_3\,a_{k3} + ... + X_k\,a_{kk} + a_{k,n} = 0
\end{aligned}\right\} \quad . \quad (4)$$

and it follows from these equations — $X_2, X_3 \ldots X_k$ and $a_{1,n}^{2,3\ldots k}$ being regarded as unknowns —

$$a_{1,n}^{2,3\ldots k} = \frac{\begin{vmatrix} a_{12} & a_{13} \ldots a_{1k} & a_{1n} \\ a_{22} & a_{23} \ldots a_{2k} & a_{2n} \\ \cdot & & \\ \cdot & & \\ a_{k2} & a_{k3} \ldots a_{kk} & a_{kn} \\ a_{22'} & a_{23} \ldots & a_{2k} \\ a_{32} & a_{33} \ldots & a_{3k} \\ \cdot & & \\ \cdot & & \\ a_{k2} & a_{k3} \ldots & a_{kk} \end{vmatrix}}{(-1)^{k-1}} = \frac{(-1)^{k-1} \triangle_n^{\substack{1,2\ldots(k-1),k \\ 2,3\ldots k,\ \ n}}}{\text{positive number}} \quad . \quad . \quad (5)$$

Therefore the sign of

$$\frac{\triangle_n^{\substack{1,2\ldots k-1,k \\ 2,3\ldots k,\ \ n}}}{a_{1,n}} \text{ agrees with the sign of } (-1)^{k-1} \frac{a_{1,n}^{2,3\ldots k}}{a_{1,n}}. \quad . \quad . \quad (6)$$

In this proof it is unessential that the points $2, 3 \ldots k$ are consecutive; they may be as well a set of points arbitrarily chosen amongst the points $2, 3 \ldots (n-1)$.

4. *The sign of the quotient* $a_{1,n}^{2,3\ldots k} : a_{1},n$. It will be shown in this paragraph that the sign of the quotient $a_{1,n}^{2,3\ldots k} : a_{1,n}$ is given by $(-1)^{k-1}$. To this end we prove the following theorem: If a beam, supported in any number of points $A_0, A_1 \ldots A_n$ of equal height, is submitted to the action of a single force P, the deflections, occurring in any span $A_i A_{i+1}$ have a uniform sign, whereas the deflections occurring in any two consecutive spans are opposite.

If we consider two consecutive segments $A_{i-1}A_i$ and $A_i A_{i+1}$ of the beam $A_o \ldots A_n$, each of them freely supported at its ends, but loaded in these points by the moments of transition M_{i-1}, M_i and M_{i+1} (see fig. 1) the MAXWELL numbers $\beta_{i-1, i} = \beta_{i, i-1}$, β_{ii}^*, β_{ii}^{**} and

Fig. 1.

$\beta_{i+1, i} = \beta_{i+1, 1}$ may be introduced. Every number represents the slope in the point indicated by the first index, caused by a unit-moment acting in the point indicated by the second index. The single and double asteriks of $\beta_{i, i}$ denote whether the number refers to the left or to the right span.

If furthermore $S_{i, i-1}$ defines the moment of the area of reduced moments of the span (i), with respect to its left support due to the forces acting on the freely supported beam $A_{i-1} A_i$ and similarly $S_{i+1, i+1}$ the moment of the area of reduced moments of the span $(i+1)$ with respect to its right support, due to the forces acting on the freely supported beam $A_i A_{i+1}$, the following well-known relation — named after CLAPEYRON — holds:

$$\beta_{i, i-1} M_{i-1} + (\beta^*_{i, i} + \beta^{**}_{i, i}) M_i + \beta_{i, i+1} M_{i+1} = \frac{S_{i, i-1}}{l_i} + \frac{S_{i+1, i+1}}{l_{i+1}}. \quad (1)$$

Expressed in the lengths l_i and l_{i+1} of the gaps, the (variable) moment of inertia I of the cross section and the modulus of elasticity E, the numbers $\beta_{i-1, i}$, $\beta^*_{i, i}$, $\beta^{**}_{i, i}$, $\beta_{i, i+1}$ have the values

$$\left. \begin{array}{l} \beta_{i, i-1} = \dfrac{1}{l_i^2} \displaystyle\int_0^{l_i} \dfrac{x_i (l_i - x_i)}{E I} \, dx_i \, ; \beta^*_{i, i} = \dfrac{1}{l_i^2} \displaystyle\int_0^{l_i} \dfrac{x_i^2 \, dx_i}{E I} \, ; \\[4mm] \beta^{**}_{i, i} = \dfrac{1}{l_{i+1}^2} \displaystyle\int_0^{l_{i+1}} \dfrac{(l_{i+1} - x_{i+1})^2 \, dx_{i+1}}{E I} \, ; \beta_{i, i+1} = \dfrac{1}{l_{i+1}^2} \displaystyle\int_0^{l_{i+1}} \dfrac{x_{i+1} (l_{i+1} - x_{i+1})}{E I} \, dx_{i+1} \end{array} \right\} \quad (2)$$

They are essentially positive and it can be proved in a similar way as in par. 3 for the numbers a_{ij}, that

$$\beta^{**}_{i, i} \, \beta^*_{i+1, i+1} - \beta_{i+1, i} \, \beta_{i, 1+i} > 0. \quad\quad\quad (3)$$

Returning now to our continuous beam $A_0 \ldots A_n$ which may be either supported or clamped at A_0 and have a series of consecutive unloaded spans (1), (2) ... (i) for each of which the quantities $S_{i, i-1}$ and $S_{i+1, i+1}$ in eq. 1 are to be put equal to zero, we find by a repeated application of (1):

$$\left. \begin{array}{l} \beta^{**}_{0, 0} M_0 + \beta_{0, 1} M_1 = 0 \\[2mm] \beta_{1, 0} M_0 + (\beta^*_{1, 1} + \beta^{**}_{1, 1}) M_1 + \beta_{1, 2} M_2 = 0 \\[2mm] \beta_{2, 1} M_1 + (\beta^*_{2, 2} + \beta^{**}_{2, 2}) M_2 + \beta_{2, 3} M_3 = 0 \\[2mm] \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{a. s. o.} \end{array} \right\} \quad (4)$$
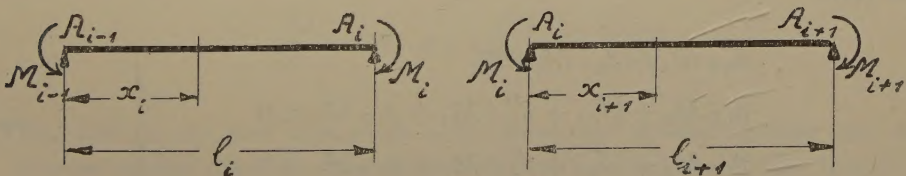
from which we deduce

$$M_0 = -\frac{\beta_{0,1}}{\beta_{0,0}^{**}} M_1 ; \quad M_1 = -\frac{\beta_{1,2} M_2}{\beta_{1,1}^{**} + \beta_{1,1}^{*} - \dfrac{\beta_{1,0}\beta_{0,1}}{\beta_{0,0}^{**}}} ; \quad M_2 = -\frac{\beta_{2,3} M_3}{\beta_{2,2}^{**} + \beta_{2,2}^{*} - \dfrac{\beta_{2,1}\beta_{1,2}}{\beta_{1,1}^{**} + \beta_{1,1}^{*} - \dfrac{\beta_{1,0}\beta_{0,1}}{\beta_{0,0}^{**}}}}$$

and in general

$$\frac{M_i}{M_{i+1}} =$$

$$= -\beta_{i,i+1} \overline{\left| \beta_{i,i}^{**} + \beta_{i,i}^{*} - \underline{\beta_{i,i-1}\beta_{i-1,i}} \right| \beta_{i-1,i-1}^{**} + \beta_{i-1,i-1}^{*} - \underline{\beta_{i-1,i-2}\beta_{i-2,i-1}} \left| \ldots \right| \beta_{1,1}^{**} + \beta_{1,1}^{*} - \dfrac{\beta_{1,0}\beta_{0,1}}{\beta_{0,0}^{**}}}$$

The fraction $M_i : M_{i+1}$ is negative and in absolute magnitude smaller than $\beta_{i,i+1} : \beta_{i,i}^{**}$. For if we denote by B_j the continual fraction

$$\beta_j = \beta_{j,j}^{*} - \underline{\beta_{j,j-1}\beta_{j-1,j}} \overline{\left| \beta_{j-1,j-1}^{**} + \beta_{j-1,j-1}^{*} - \underline{\beta_{j-1,j-2}\beta_{j-2,j-1}} \left| \ldots \right| \beta_{1,1}^{**} + \beta_{1,1}^{*} - \dfrac{\beta_{1,0}\beta_{0,1}}{\beta_{0,0}^{**}} \right.}$$

we know by (3) that

$$B_1 \equiv \frac{\beta_{1,1}^{*}\beta_{0,0}^{**} - \beta_{1,0}\beta_{0,1}}{\beta_{0,0}^{**}}$$

is essentially positive. Therefore

$$B_2 \equiv \beta_{2,2}^{*} - \frac{\beta_{2,1}\beta_{1,2}}{\beta_{1,1}^{**} + B_1} > \beta_{2,2}^{*} - \frac{\beta_{2,1}\beta_{1,2}}{\beta_{1,1}^{**}} = \frac{\beta_{2,2}^{*}\beta_{1,1}^{**} - \beta_{2,1}\beta_{1,2}}{\beta_{1,1}^{**}}.$$

The latter fraction again is positive so that $B_2 > 0$. Proceeding in this way we find at last

$$\frac{M_i}{M_{i+1}} = -\frac{\beta_{i,i+1}}{\beta_{i,i}^{**} + B_i}$$

where $\beta_{i,i+1}$, $\beta_{i,i}^{**}$ and B_i all represent positive numbers. Consequently

$$\frac{M_i}{M_{i+1}} < 0 \quad \text{and} \quad \left| \frac{M_i}{M_{i+1}} \right| < \frac{\beta_{i,i+1}}{\beta_{i,i}^{**}}.$$

The first of the two statements still leaves the possibility that the deflection of the span $A_i A_{i+1}$ should change its sign over the length l_i (obviously such change is excluded if M_i and M_{i+1} were both positive

or negative). The second one however makes it certain that such change cannot occur. For if we try to find the moment M_i^* (M_{i+1} regarded as
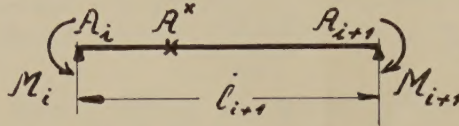

Fig. 2.

a given positive moment) that would neutralize the slope caused at A_i by M_{i+1}, (see fig. 2), the condition

$$M_{i+1}\,\beta_{i,i+1} + M_i^*\,\beta_{i,i}^{**} = 0$$

must be fulfilled, and therefore

$$M_i^* = -\frac{\beta_{i,i+1}}{\beta_{i,i}^{**}}\,M_{i+1} \quad \text{or} \quad \frac{M_i^*}{M_{i+1}} = -\frac{\beta_{i,i+1}}{\beta_{i,i}^{**}}.$$

If $\left|\dfrac{M_i}{M_{i+1}}\right|$ should be $> \dfrac{\beta_{i,i+1}}{\beta_{i,i}^{**}}$ a point A^* with zero deflection would arise at the right side of A_i. In our case however, where $\left|\dfrac{M_i}{M_{i+1}}\right| < \dfrac{\beta_{i,i+1}}{\beta_{i,i}^{**}}$ neither such point nor a zero-slope at the point A_i can appear.

Consequently the deflection of the beam changes its sign at every support and keeps it in every span. This being fixed the sign of the quotient $a_{1,n}^{2,3\ldots k} : a_{1n}$ can easily be found. For indeed the sign of a_{1n} corresponds with the sign of $(-1)^p$, p being the number of real shaft-supports between the points 1 and n. If, however, at the points $2,3\ldots k$ a set of $(k-1)$ supports is introduced the sign of $a_{1,n}^{2,3\ldots k}$ corresponds with the sign of $(-1)^{p+(k-1)}$. Therefore the sign of $a_{1,n}^{2,3\ldots k} : a_{1,n}$ is given by $(-1)^{k-1}$, and consequently the sign of the quotient: $\triangle_n^{2,3\ldots\ k\ \ n} : a_{1,n}$ which we knew already to agree with the sign of $(-1)^{k-1}\,a_{1,n}^{2,3\ldots k} : a_{1,n}$ (compare (6, 3) is positive for all values $2 \leqq k \leqq (n-1)$.

5. *The unequality of the critical angular speeds.* With the notation introduced in the beginning of par. 3 the equations (2, 4a) and (2, 4b) can be written as follows:

$$\left.\begin{array}{l} z^{n-1} - \left[\displaystyle\sum_{i=1}^{n-1} m_i\,\triangle_n^i\right]z^{n-2} + \left[\displaystyle\sum_{\substack{i=1 \\ i<j}}^{n-1}\sum_{j=1}^{n-1} m_i\,m_j\,\triangle_n^{i,j}\right]z^{n-3} - \\[4mm] - \left[\displaystyle\sum_{\substack{i=1 \\ i<j<k}}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} m_i\,m_j\,m_k\,\triangle_n^{i,j,k}\right]z^{n-4} + \ldots + (-1)^{n-2}m_1\,m_2\ldots m_{n-1}\triangle_{n-1} = 0 \end{array}\right\} \quad (1)$$

$$a_{1n} z^{n-2} + \left[\sum_{i=2}^{n-1} m_i \triangle_n^{1,i} \right] z^{n-3} + \left[\sum_{i=2}^{n-1} \sum_{\substack{j=2 \\ i<j}}^{n-1} m_i\, m_j \triangle_n^{1,i,j} \right] z^{n-4} +$$

$$+ \left[\sum_{i=2}^{n-1} \sum_{\substack{j=2 \\ i<j<k}}^{n-1} \sum_{k=2}^{n-1} m_i\, m_j\, m_k \triangle_n^{1,i,j,k} \right] z^{n-5} + \ldots +$$

$$+ m_2\, m_3 \ldots m_{n-1} \triangle_n^{1,2\ldots(n-1)}_{2,3\ldots n} = 0 \qquad \cdots \quad (2)$$

As to equation (1) — or (2, 4a) — it is well-known, that the set of functions

$$S_{n-1},\ S_{n-2},\ S_{n-3},\ \ldots S_1,\ +1 \quad \cdots\ \cdots\ \cdots \quad (3)$$

constitutes a so-called set of STURM-functions. If therefore two arbitrary values $z = p$ and $z = q\ (p > q)$ are substituted in the functions (3), the loss of variations indicates the number of real roots between p and q. If for p and q the values ∞ and 0 are taken we find, with $z = \infty$ the following signs

$$(-1)^{n-1}\quad (-1)^{n-2}\ \ldots (-1)^1\quad +1$$

that is: $(n-1)$ variations.

On the contrary, if $z = 0$ is substituted, the values of $S_{n-1}, S_{n-2} \ldots (+1)$ are represented by the essential positive numbers

$$m_1\, m_2 \ldots m_{n-1} \triangle_{n-1},\ m_1\, m_2 \ldots m_{n-2} \triangle_{n-2},\ \ldots,\ m_1 \triangle_1,\ +1$$

which have no variations of sign at all.

Equation (1) therefore has $(n-1)$ *positive* roots. As to equation (2), if it be divided by the coefficient a_{1m} it follows from par. (4) that all coefficients of z^{n-2}, z^{n-3} a. s. o. are essentially positive and therefore this equation has *no* positive roots. Consequently common roots of (1) and (2) do not exist and this involves (comp. par. 2) the inequality of all critical speeds.

6. *Concluding remark.* It must be remarked that it is a fundamental feature of the foregoing proof, that all supports are rigid and that at these supports no zero-slope can occur if the shaft is loaded by a single force. Therefore supports — terminal ones eventually excepted — which should have a "clamping" effect, must be excluded. In fact it is seen at once that the shaft, of constant diameter, represented in fig. 3, for which
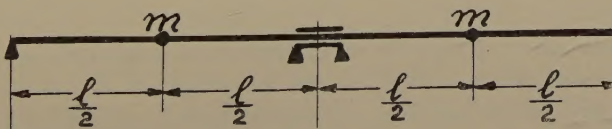

Fig. 3.

$a_{12} = 0$, only has one single critical speed which is identical with the critical speed of the shaft represented in fig. 4.
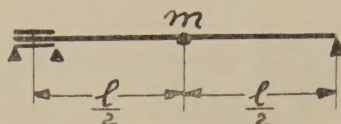

Fig. 4.

Elastic supports too must be excluded. For it is an easy matter — supposing that the middle support of the shaft represented in fig. 5 is
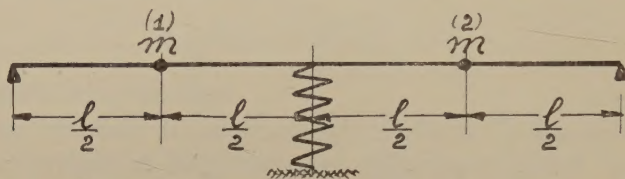

Fig. 5.

an elastic one with the elastic constant c — to calculate c in such a way that two equal critical speeds occur. Indeed, the condition for equal critical speeds

$$\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2 = 0$$

which follows from

$$\begin{vmatrix} m\,a_{11} - z & m\,a_{12} \\ m\,a_{21} & m\,a_{22} - z \end{vmatrix} = 0$$

requires $a_{12} = 0$. The constant c therefore must be chosen in such a way, that a load placed in (1) produces a zero-deflection in (2). If the shaft has a constant cross section we find $c = \dfrac{28}{3}\,\dfrac{EI}{l^3}$.

Therefore the theorem proved in this paper only holds for rigid, non-clamping supports. An exception has to be made for terminal supports, which must be rigid, but may be clamping.

Hydrodynamics. — *On the application of statistical mechanics to the theory of turbulent fluid motion. A hypothesis which can serve as a basis for a statistical treatment of some mathematical model systems. II.* By J. M. BURGERS. (Mededeeling N⁰. 39 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft). *)

9. *Discussion of the result obtained in Part I.* — The result which has been arrived at in Part I can be formulated as follows:

(a) The introduction of a regular point lattice with a finite spacing $\delta = \beta \nu$, instead of a continuous field, into the phase space for the variables ξ_n and η_n, has made it possible to derive formulae for the average statistical values of ξ_n^2 and η_n^2 which do not lead to a divergent expression for the total dissipation of energy. (b) These formulae present a similar course as the values of ξ_n^2 and η_n^2 which were estimated from an investigation of the solutions of the differential equations defining the mathematical model system. (c) The average values of ξ_n^2 and η_n^2 are directly connected with the magnitude of the "exterior force" P — compare eq. (28) —, in such a way that the "exterior force" practically is balanced by reactions deriving from the first few components of the secondary motion only. (d) It appeared possible to obtain a definite relation between P and the "velocity" U of the principal motion — compare eq. (30) — in which no other unknown constant appeared besides the factor β; the relation leads to a quadratic "resistance law".

The following observation still is of interest: From the calculations given in 8 it will be seen that we may write:

$$U = 6 \nu N_s \quad \ldots \ldots \ldots \ldots \quad (31)$$

where N_s is the quantity defined by:

$$N_s = \frac{\Sigma \, n^2 \, (\overline{\xi_n^2} + \overline{\eta_n^2})}{12 \, P/\pi^2} \quad \ldots \ldots \ldots \quad (32)$$

We may say that N_s measures the *"effective length"* of the spectrum of elementary components of the secondary motion; *i. e.* N_s is the number of components which should be counted in making up the total dissipation, if eq. (28) applied to all values of n until $n = N_s$, while beyond that value the spectrum would be suddenly cut off. The calculations given in 8 show that approximately: $N_s \simeq 2{,}7\,N_1$. (It would be important to find a more direct and more accurate way for calculating N_s than the procedure employed in 8).

It seems probable that equation (31) is representative of a relation of general character in systems of this kind. The fact that we have obtained a quadratic "resistance law" is a consequence of the circumstance that for the system considered N_s is proportional to $\sqrt{P}$.

The introduction of the point lattice into the phase space thus in a certain way has brought an answer to the problem stated in a previous communication on this subject [13]).

It should be understood that the introduction of this point lattice is no more than a mathematical artifice, applied in order to take notice of the peculiar properties of the system which are due to the simultaneous presence of non linear terms and of frictional terms in the equations. The value of the coefficient β, occurring in the expression for the lattice spacing δ, should be obtained from a more rigorous treatment of this subject; it is not to be considered as an undetermined quantity, which at the end of the calculations might be adjusted at will.

10. Apart from the introduction of the point lattice, another matter attracts notice, *viz.* the form of the dissipation equation which has been chosen as the basis for the statistical calculations. As will be seen from eq. (7) these calculations have been built upon the assumption that the average dissipation of energy in unit of time by the secondary motion should have a fixed value, given as PU. This assumption has the great advantage that it makes the calculations as simple and direct as possible. It differs, however, from the elimination procedures which had been applied in previous papers, where either the mean value of the velocity or that of the exterior force (the pressure gradient) was considered as being given.

It can be tried to apply a similar procedure to the system under consideration, but it is found that this leads to a rather unprobable result. We take the case where P is given, and consider U as a variable quantity, which may have the value U_m in the instantaneous state numbered m. When we write:

$$E_m = \tfrac{1}{2} \sum_n (\xi_{nm}^2 + \eta_{nm}^2) \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (33)$$

[13]) Compare: On the application of statistical mechanics to the theory of turbulent fluid motion, Part. VII, these Proceedings **36**, 628 (1933).

then instead of eq. (7) the following two equations must be fulfilled by every sequence of instantaneous states [14]:

$$\sum_m f_{ms} (P - \nu U_m - E_m) = 0 \quad \ldots \quad (34a)$$

$$\sum_m f_{ms} (P U_m - \nu U_m^2 - \varepsilon_m) = 0 \quad \ldots \quad (34b)$$

The method of the partition function cannot be directly applied in this case, as the expression $P U_m - \nu U_m^2 - \varepsilon_m$ is not always of the same sign. However, the application of the methods for calculating the "most probable statistical state" gives:

$$\overline{f_m} = C \, e^{\lambda_1 (P - \nu U_m - E_m) + \lambda_2 (P U_m - \nu U_m^2 - \varepsilon_m)} \quad \ldots \quad (35)$$

where λ_1 and λ_2 are two positive constants (λ_2 is related to $- \ln \sigma$ in sections 7—8).

As U_m may have all positive and negative values, it is found that the average value of U is equal to that value of U_m which makes $\overline{f_m}$ a maximum. Hence:

$$\overline{U} = (\lambda_2 P - \lambda_1 \nu) / 2 \lambda_2 \nu \, . \quad \ldots \quad (36)$$

The average values of ξ_n^2 and η_n^2 are given by a similar formula as (21), in which now:

$$\theta_n = e^{-(\lambda_1 + \lambda_2 \nu \pi^2 n^2) \delta^2 / 2} \, . \quad \ldots \quad (37)$$

For not too large values of n this will lead to the following expression for $\overline{\xi_n^2}$:

$$\overline{\xi_n^2} \simeq \frac{1}{\lambda_1 + \lambda_2 \nu \pi^2 n^2} \cdot \quad \ldots \quad (38)$$

Now from (36) it will be seen that $\overline{U}$ will become small compared with P (e.g. will become of the order $\sqrt{P}$) only then, when $\lambda_1 \nu \simeq \lambda_2 P$. In that case, however, λ_1 will greatly exceed the value of $\lambda_2 \nu \pi^2 n^2$ for values of n which are smaller than an amount of the order $\sqrt{P}/\pi \nu$. This would lead to the result that $\overline{\xi_n^2}$ for such values of n should be independent of n, instead of being proportional to n^{-2}. I am inclined to consider such a result as unprobable, as most observations on turbulent motion give the impression that the components of a more coarse pattern (components of long wave length, large vortices) are preponderant in the balance of forces.

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[14] Compare the paper mentioned in footnote 1 (see p. 936 above), p. 44, eqs. (20.5) and (20.10).

11. The problem next presents itself whether the introduction of a point lattice into the phase space also may serve to bring to a more successful end the investigations on turbulence undertaken in some earlier papers [15]). It must be mentioned beforehand that these investigations had been based upon the assumption of a twodimensional field of flow, so that they cannot be considered as referring to the actual pattern of turbulent fluid motion in a tube or a channel, which pattern is three-dimensional in nature (it must be supposed that in this motion vortices with their axes parallel to the general direction of the flow play a preponderant role). Hence the twodimensional system should be considered as merely representing another model system.

We abandon the procedures for the elimination of the pressure gradient formerly employed; neither can we calculate the principal motion by equating $\mu dU/dy$ to the difference between the value of the total shearing force τ found from the pressure gradient and the value of $-\overline{\varrho uv}$ derived from the statistical calculations. As the difference between τ and $-\overline{\varrho uv}$ necessarily is small, it must be considered that this method, which was applied in some previous papers [16]), cannot guarantee a sufficiently accurate result. Hence a different procedure should be found for determining the principal motion, and it is this point which now brings the main difficulty into the problem. An attempt perhaps may be made along the following lines.

Provisionally we assume that the principal motion $U(y)$ is given. The secondary motion at any instant can be described by giving the stream function ψ, so that:

$$u = \partial\psi/\partial y, \qquad v = -\partial\psi/\partial x; \qquad \zeta = -\triangle\psi \quad . \quad . \quad . \quad (39)$$

We suppose that ψ can be resolved into a series of elementary components of the form:

$$\chi_\mathrm{I}(y) \cos(ax + \varepsilon) + \chi_\mathrm{II}(y) \sin(ax + \varepsilon),$$

in which the functions $\chi_\mathrm{I}, \chi_\mathrm{II}$ are chosen in such a way that these elementary components give a positive maximum value to the fraction:

$$\frac{1}{Vb} \frac{-\iint dx\, dy\, uv\, \dfrac{dU}{dy}}{\iint dx\, dy\, \zeta^2} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

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[15]) On the application of statistical mechanics to the theory of turbulent fluid motion; these Proceedings **32**, 414, 634, 818 (1929); **36**, 276, 390, 487, 620 (1933).

[16]) Compare: On the resistance experienced by a fluid in turbulent motion, these Proceedings **26**, 582 (1923), and the papers mentioned in the preceding footnote. The method has been criticized already in Part VII of these papers, these Proceedings **36**, 620 (under II) (1933).

Here V represents the mean velocity of the principal motion over a cross section of the channel, while b is the breadth of the channel; these factors have been introduced in order that the fraction shall become a nondimensional quantity.

When we write: $\chi = \chi_\mathrm{I} + i\,\chi_\mathrm{II}$, it is found that χ must be a solution of the equation:

$$\chi^\mathrm{IV} - 2\,a^2\,\chi'' + a^4\,\chi - i a \,\frac{\varLambda}{Vb}\left(\frac{dU}{dy}\,\chi' + \frac{1}{2}\,\frac{d^2U}{dy^2}\,\chi\right) = 0. \quad . \quad (41)$$

As boundary conditions for χ we prescribe that both χ and χ' shall be zero at the walls of the channel; then for every value of a eq. (41) will admit solutions only when the parameter $\varLambda$ is equal to a characteristic value of the equation; these characteristic values, which are nondimensional and which are dependent upon the magnitude of a, will be denoted by $\varLambda_{an}$. For any solution χ_{an} of (41) we have:

$$-\overline{(u_{an}\,v_{an})} = \tfrac{1}{2}\,a\,(\chi'_{\mathrm{I}an}\,\chi_{\mathrm{II}an} - \chi_{\mathrm{I}an}\,\chi'_{\mathrm{II}an})$$

$$\overline{(\zeta_{an})^2} = \tfrac{1}{2}\,(\chi''_{\mathrm{I}an} - a^2\,\chi_{\mathrm{I}an})^2 + \tfrac{1}{2}\,(\chi''_{\mathrm{II}an} - a^2\,\chi_{\mathrm{II}an})^2,$$

where the mean values are taken with respect to x.

We now prescribe as a normalizing condition for the functions $\chi_{\mathrm{I}an}$, $\chi_{\mathrm{II}an}$:

$$-\frac{1}{V}\int dy\,\overline{(u_{an}\,v_{an})}\,\frac{dU}{dy} = 1 \quad . \quad . \quad . \quad . \quad . \quad (42a)$$

It is then found that:

$$b\int dy\,\overline{(\zeta_{an})^2} = \varLambda_{an} \quad . \quad . \quad . \quad . \quad . \quad (42b)$$

It will be seen that the characteristic values $\varLambda_{an}$ necessarily are positive. — The normalizing condition (42a) ensures that into the development of the function ψ only such elementary components are admitted which can derive energy from the primary motion. A further consequence of (42a) is that the functions $\chi_{\mathrm{I}an}$, $\chi_{\mathrm{II}an}$ will have the dimensions of a length.

12. As a general expression for the stream function of the secondary motion we now write:

$$\psi = \sum_{an}[\xi_{an}\,(\chi_{\mathrm{I}an}\cos ax + \chi_{\mathrm{II}an}\sin ax) + \eta_{an}\,(-\chi_{\mathrm{I}an}\sin ax + \chi_{\mathrm{II}an}\cos ax)] \quad (43)$$

The coefficients ξ_{an}, η_{an} have the dimensions of a velocity. In general they must be considered as functions of the time; a particular instantaneous state m of the secondary motion, however, can be specified by giving the values ξ_{anm}, η_{anm} of the coefficients for that state. It is usually

assumed that the parameter a may run through a series of equidistant values of the form $2\pi h/L$, where L is the length of the channel, while h is a positive integer.

In consequence of the choice adopted for the functions χ the dissipation integral (taken per unit of length of the channel) for the state of motion m can be expressed in the following form:

$$\varepsilon_m = \frac{\mu}{L} \iint dx\, dy\, (\zeta^2)_m = \frac{\mu}{b} \sum_{\alpha n} \Lambda_{\alpha n} (\xi^2_{\alpha n m} + \eta^2_{\alpha n m}) \quad . \quad . \quad (44)$$

The average statistical state of the system now may be calculated upon the assumption that the average dissipation of energy in unit time shall have a given value. In equation (7) we substitute the expression (44) for ε_m, and as before we suppose that the coefficients $\xi_{\alpha n m}$ and $\eta_{\alpha n m}$ only can take values which are integer multiples of a "threshold value" δ. This "threshold value" must have the dimensions of a velocity; judging from the equations of motion it probably will be of the order $\mu/\varrho b$.

Applying the methods described in sections 4—7 it is found that the statistical mean values of $\xi^2_{\alpha n}$ and $\eta^2_{\alpha n}$ again will be given by eq. (21), in which θ_n must be replaced by $\theta_{\alpha n} = \sigma^{\mu \Lambda_{\alpha n} \partial^2/b}$. It is convenient to write: $\sigma^{\mu \partial^2/b} = e^{-1/C}$, so that:

$$\theta_{\alpha n} = e^{-\Lambda_{\alpha n}/C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Then for moderate values of a and n we may expect:

$$\overline{\xi^2_{\alpha n}} = \overline{\eta^2_{\alpha n}} \simeq \frac{\delta^2}{-2\ln \theta_{\alpha n}} = \frac{\delta^2 C}{2 \Lambda_{\alpha n}} \quad . \quad . \quad . \quad . \quad (46)$$

whereas for large values of a and n:

$$\overline{\xi^2_{\alpha n}} = \overline{\eta^2_{\alpha n}} \simeq 2\,\delta^2\,\theta_{\alpha n} = 2\,\delta^2\,e^{-\Lambda_{\alpha n}/C} \quad . \quad . \quad . \quad . \quad (47)$$

(compare eqs. (24b), (24a) of Part I). The transition between the two types of approximation again will be determined by $\theta_{\alpha n} = 0.25$, i. e.:

$$\Lambda_{\alpha n} = C \ln 4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Equation (48) provides us with an indication concerning the limit of the spectrum of elementary components; in this respect it plays a similar role as the condition $(\nu \pi^2\, N_1^2\, \delta^2/2)\,.\, \ln \sigma = \ln 0.25$ applied in 8. The actual number N_s of components which should be counted in making up the total dissipation, assuming as the dissipation per component the value:

$$(\mu/b)\, \Lambda_{\alpha n} (\overline{\xi^2_{\alpha n}} + \overline{\eta^2_{\alpha n}}) = (\mu/b)\, \delta^2\, C \quad . \quad . \quad . \quad . \quad (49)$$

still will be larger than the number of components which satisfy eq. (48), as can be judged from the relation between N_s and N_1 found for the original model system.

13. The formulae obtained in the preceding section lead to the following result: When the characteristic values $\varLambda_{\alpha n}$ of equation (41) have been calculated, and when values have been assumed for δ and for C, the average statistical values of $\xi^2_{\alpha n}$ and $\eta^2_{\alpha n}$ can be found from (46) and (47). The average statistical state of the secondary motion then is known, and it is possible to calculate the average value of $-\overline{uv}$ for this state as a function of y. Now this value practically must be equal to the value of the shearing force τ which can be found from the pressure gradient. When the calculations are started with an arbitrary expression for dU/dy in (41), it is not at all certain that this condition shall be fulfilled. It seems probable, therefore, that this condition can be satisfied only with a particular choice of dU/dy. If this should prove to be the case, then this particular function dU/dy might be considered as giving that distribution of the principal motion over the channel which apparently is in equilibrium with the average statistical state of the secondary motion.

The application of this condition to find the value of dU/dy will be a difficult matter, although it is not impossible that approximate calculations already may give much help.

Another point, however, should be noted. In eq. (41) we have to do with the quantity $\dfrac{1}{V}\dfrac{dU}{dy}$; hence the application of the condition cannot give the absolute values of the velocity, but only the relative distribution. The absolute value of V should be found from the dissipation condition, in a similar way as in the original model system the value of U was obtained from eq. (29) or from eq. (31). We must expect, therefore, that the value of V again will depend upon the number N_s of elementary components which must be counted in making up the total dissipation. Some rough estimates which were made in order to get a provisional idea pointed to a new difficulty, connected with the circumstance that the spectrum now is determined by two parameters, α and n: the value estimated for N_s was of a higher order than was expected. As it is no easy matter to obtain a reliable picture of the influence of the expression adopted for dU/dy upon the values of the characteristic numbers $\varLambda_{\alpha n}$, it has not been possible as yet to arrive at definite results.

Mathematics. — *Ueber algebraische Systeme von partiellen Differentialgleichungen erster Ordnung.* II. *Gleichungen mit einer Unbekannten.*

Beweis des Existenz theorems.

Von J. A. Schouten und W. van der Kulk.

(Communicated at the meeting of October 26, 1940.)

6. *Regularität der Gleichungen einer irreduzibelen algebraischen Mannigfaltigkeit in einem einzigen Punkt.*

Bekanntlich nennt man d die Dimension einer irreduzibelen algebraischen Mannigfaltigkeit $\mathfrak{M}$, wenn d der $n-1$ Verhältnisse der Koordinaten $w_1 \ldots, w_n$ eines allgemeinen Punktes algebraisch unabhängig sind i.b. auf den Körper der meromorphen Funktionen der $x^\varkappa$ und die übrigen $n-1-d$ sich algebraisch i.b. auf diesen Körper in diese d ausdrücken lassen [1]). Bilden die Tensoren $P^{\mu\lambda_1 \ldots \lambda_{a_i}}_i$, $i = 1, \ldots, N$ eine Basis des zugehörigen Ideals von $\mathfrak{M}$, so sind

$$G_i(x^\varkappa, w_\lambda) \overset{\text{def}}{=} P^{\mu\lambda_1 \ldots \lambda_{a_i}}_i \, w_\mu \, w_{\lambda_1} \ldots w_{\lambda_{a_i}} = 0; \quad i = 1, \ldots, N \quad \varkappa, \lambda = 1, \ldots, n \quad (1)$$

die Gleichungen von $\mathfrak{M}$. Ist nun $w_\lambda = \overset{0}{w}_\lambda$ in $x^\varkappa = \overset{0}{x}{}^\varkappa$ ein zu $\mathfrak{M}$ gehöriger Vektor und schreiben wir

$$G^\varkappa_i(w_\lambda) \overset{\text{def}}{=} \left(\frac{\partial G_i}{\partial w_\varkappa} \right)_{x^\mu = \overset{0}{x}{}^\mu} \quad ; \quad i = 1, \ldots, N, \quad \ldots \ldots \quad (2)$$

so bestimmen die Gleichungen

$$\overset{0}{G}{}^\varkappa_i(w_\lambda) \, w_\varkappa = 0 \quad ; \quad i = 1, \ldots, N \ . \ . \ . \ . \ . \ . \quad (3)$$

in der lokalen P_{n-1} von $\overset{0}{x}{}^\varkappa$ eine ebene Mannigfaltigkeit. Ist die Dimension derselben d, d.h. gibt es unter den N Hyperebenen (3) genau $n-1-d$ linear unabhängige, so nennen wir sie den *Tangentialraum* von $\mathfrak{M}$ in $\overset{0}{x}{}^\varkappa$, $\overset{0}{w}_\lambda$ [2]).

Wir beweisen jetzt den

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[1]) Vgl. E. a. G., S 112.
[2]) Vgl. E. a. G., S. 171.

Satz: *Besitzt eine d-dimensionale irreduzibele algebraische Mannigfaltigkeit $\mathfrak{M}$ in $\overset{0}{x^\varkappa}$, $\overset{0}{w_\lambda}$ einen Tangentialraum, so lassen sich $n-1-d$ der $n-1$ Verhältnisse der w_λ in der Umgebung von $\overset{0}{x^\varkappa}$, $\overset{0}{w_\lambda}$ schreiben als analytische Funktionen der $x^\varkappa$ und der d übrigen.*

Beweis: Da ein Tangentialraum existiert, hat die Matrix von $\overset{0}{G_i^\varkappa(w_\lambda)}$ den Rang $n-1-d$. Durch Umnumerieren kann man also stets erreichen, dass die Determinante von $\overset{0}{G_j^\gamma(w_\lambda)}$; $\gamma=d+2,\ldots,n$; $j=1,\ldots,n-1-d$ nicht verschwindet. Aus der Homogenität der Gleichungen (1) in w_λ folgt

$$\overset{0}{w_\lambda}\,\overset{0}{G_j^\lambda(w_\mu)}=(a_j+1)\,G_j\,(\overset{0}{x^\varkappa},\overset{0}{w_\lambda})=0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Wären nun $\overset{0}{w_1},\ldots,\overset{0}{w_{d+1}}$ alle gleich Null, so würde aus (4) folgen

$$\overset{0}{w_\gamma}\,\overset{0}{G_j^\gamma(w_\lambda)}=0 \; ; \; \gamma=d+2,\ldots,n \; ; \; j=1,\ldots,n-1-d \; . \quad . \quad (5)$$

und dies ist nicht möglich, da $\overset{0}{G_j^\gamma(w_\lambda)}$ den Rang $n-1-d$ hat und $\overset{0}{w_{d+2}},\ldots,\overset{0}{w_n}$ nicht alle zugleich Null sein können da sonst w_λ Null wäre. Durch Umnumerieren kann man also stets erreichen, dass *gleichzeitig* $\overset{0}{w_1}\neq 0$ ist und Det $(\overset{0}{G_j^\gamma(w_\lambda)})\neq 0$ ist.

Wir schreiben jetzt $\omega_2,\ldots,\omega_n$ für die Verhältnisse $\frac{w_2}{w_1},\ldots,\frac{w_n}{w_1}$ und

$$H_j\,(x^\varkappa,\omega_\beta)\overset{\text{def}}{=}G_j\,(x^\varkappa,1,\omega_\beta)\,\overline{;}\;\beta=2,\ldots,n\;;\;j=1,\ldots,n-1-d. \quad (6)$$

Dann ist

$$\frac{\partial H_j}{\partial\omega_\beta}=\frac{1}{w^{a_j}}\frac{\partial G_j}{\partial w_\beta}\;;\;\beta=2,\ldots,n\;\;j=1,\ldots,n-1-d. \quad . \quad (7)$$

Daraus folgt, dass in $\overset{0}{x^\varkappa}$, Det $\left(\frac{\partial H_j}{\partial\omega_\gamma}\right)$ für $\omega_\beta=\frac{\overset{0}{w_\beta}}{\overset{0}{w_1}}$ nicht verschwindet. Aus den Gleichungen

$$H_j\,(x^\varkappa,\omega_\beta)=0\;;\;\beta=2,\ldots,n\;;\;j=1,\ldots,n-1-d \quad . \quad (8)$$

lassen sich also nach der Theorie der Funktionaldeterminanten $\omega_{d+2},\ldots,\omega_n$ in einer Umgebung von $\overset{0}{x^\varkappa}$, $\overset{0}{\omega_\beta}$ als analytische Funktionen von $x^\varkappa,\omega_2,\ldots,\omega_{d+1}$ lösen

$$\left.\begin{array}{l}\omega_{d+2}=f_{d+2}\,(x^\varkappa,\omega_\varepsilon)\\ \quad\vdots \qquad\quad \vdots \\ \omega_n\;\;=f_n\;\;(x^\varkappa,\omega_\varepsilon)\end{array}\right\}\left.\begin{array}{l}\varepsilon=2,\ldots,d+1\\ \varkappa=1,\ldots,n\end{array}\right\} \quad . \quad . \quad . \quad (9)$$

Diese Gleichungen stellen nun einen allgemeinen Punkt einer irreduzibelen d-dimensionalen algebraischen Mannigfaltigkeit $\mathfrak{M}'$ dar [3]). Die Punkte von $\mathfrak{M}$ erfüllen aber (8), also auch (9), d.h. es ist $\mathfrak{M} \subseteq \mathfrak{M}'$. Da aber $\mathfrak{M}$ ebenfalls irreduzibel und d-dimensional ist, ist $\mathfrak{M} = \mathfrak{M}'$ [3a]). $\mathfrak{M}$ lässt sich also in der Umgebung von $\overset{0}{x^{\varkappa}}, \overset{0}{w_{\beta}}$ durch ein System von Gleichungen der Form (9) mit in dieser Umgebung analytischen Funktionen der $x^{\varkappa}$ und ω_{ε}, $\varepsilon = 2, \ldots, d+1$, darstellen, w.z.b.w.

In jedem Punkte $\overset{0}{x^{\varkappa}}, \overset{0}{w_{\lambda}}$, wo ein Tangentialraum existiert, lässt sich aus $G_j^{\varkappa}(\overset{0}{w_{\lambda}})$ (bis auf einen skalaren Faktor) ein einfacher kontravarianter $(n{-}1{-}d)$-Vektor konstruieren

$$u^{\varkappa_{d+2}\cdots \varkappa_n} = \{\, G_1^{[\varkappa_{d+2}} \cdots G_{n-1-d}^{\varkappa_n]}\,\}_{w_{\lambda} = \overset{0}{w_{\lambda}}}, \quad . \quad . \quad . \quad (10)$$

den wir den Tangential $(n{-}1{-}d)$-Vektor in $\overset{0}{x^{\varkappa}}, \overset{0}{w_{\lambda}}$ nennen. Wir können das erhaltene Resultat also auch folgendermassen aussprechen:

Besitzt $\mathfrak{M}$ *in* $\overset{0}{x^{\varkappa}}, \overset{0}{w_{\lambda}}$ *einen Tangentialraum, und ist daselbst die Bestimmungszahl* $u^{d+2\cdots n}$ *ungleich Null, so sind* $\overset{0}{w_1}, \ldots, \overset{0}{w_{d+1}}$ *nicht alle zugleich Null. Ist z.B.* $w_1 \neq 0$, *so lässt sich* $\mathfrak{M}$ *in der Umgebung von* $\overset{0}{x^{\varkappa}}, \overset{0}{w_{\lambda}}$ *durch (9) darstellen.*

Es ist noch zu bemerken, dass $\mathfrak{M}$ in der Umgebung von $\overset{0}{x^{\varkappa}}, \overset{0}{w_{\lambda}}$ der vollständige Durchschnitt ist der $n{-}d{-}1$ algebraischen Hyperflächen (8), deren Tangentialräume in $\overset{0}{x^{\varkappa}}, \overset{0}{w_{\lambda}}$ linear unabhängig sind, da $\dfrac{\partial H_j}{\partial w_{\beta}}$ den höchsten Rang $n{-}d{-}1$ hat.

7. Das Existenztheorem.

Wir beweisen jetzt das

Existenztheorem: Ist $\mathfrak{M}$ *eine* d-*dimensionale vollständige irreduzibele algebraische Mannigfaltigkeit,* $\overset{0}{x^{\varkappa}}$ *ein Punkt wo auch die lokale zu* $\mathfrak{M}$ *gehörige algebraische Mannigfaltigkeit irreduzibel ist und* $\overset{0}{w_{\lambda}}$ *ein zu* $\mathfrak{M}$ *gehöriger Vektor in* $\overset{0}{x^{\varkappa}}$, *während* $\mathfrak{M}$ *in* $\overset{0}{x^{\varkappa}}, \overset{0}{w_{\lambda}}$ *einen Tangentialraum besitzt, so existiert mindestens ein in der Umgebung von* $\overset{0}{x^{\varkappa}}$ *analytisches zu* $\mathfrak{M}$ *gehöriges Gradientfeld* $\partial_{\lambda} s$, *dessen Feldwert in* $\overset{0}{x^{\varkappa}}$ *mit* $\overset{0}{w_{\lambda}}$ *zusammenfällt.*

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[3]) Vgl. E. a. G. § 29.
[3a]) Vgl. E. a. G. S. 112.

Beweis.

Das Koordinatensystem sei so gewäht, dass der Massvektor $\overset{n}{e_\lambda}$ nicht zu $\mathfrak{M}$ gehört, in $\overset{0}{x^\varkappa}$ weder mit w_λ zusammenfällt noch in dem zu $\overset{0}{x^\nu}, w_\lambda$ gehörigen Tangentialraum liegt, während für $d < n-2$ ausserdem die in $\overset{0}{x^\varkappa}$ durch $\overset{0}{w_\lambda}$ und $\overset{n}{e_\lambda}$ gelegte Verbindungsgerade ausser $\overset{0}{w_\lambda}$ keinen einzigen Punkt mit $\mathfrak{M}$ gemeinschaftlich hat. Für den Tangential-$(n-1-d)$-Vektor $u^{\varkappa d+2\cdots\varkappa_n}$ in $\overset{0}{x^\varkappa}, w_\lambda$ gilt also

$$u^{\varkappa d+2\cdots\varkappa_n}\,\overset{n}{e_{\varkappa_n}}\neq 0 \quad \ldots \ldots \ldots \quad (11)$$

d.h. es ist wenigstens eine Bestimmungszahl mit einem Index n ungleich Null. Durch Umnumerieren kann man also erreichen dass $u^{d+2\cdots n}$ ungleich Null ist. Nach dem im vorigen Paragrafen erhaltenen Resultat kann man durch weiteres Umnumerieren erreichen dass ausserdem $\overset{0}{w_1}$ ungleich Null ist. Nach dem daselbst bewiesenen Satz lässt sich $\mathfrak{M}$ in der Umgebung von $x^\varkappa, w_\lambda$ darstellen durch Gleichungen von der Form (9) oder auch

$$\left.
\begin{aligned}
w_{d+2} &= w_1 f_{d+2}\left(x^\varkappa, \frac{w_\varepsilon}{w_1}\right) \overset{\text{def}}{=} \varphi_{d+2}\left(x^\varkappa, w_1, \ldots, w_{d+1}\right) \\
&\;\vdots \\
w_n &= w_1 f_n\left(x^\nu, \frac{w_\varepsilon}{w_1}\right) \overset{\text{def}}{=} \varphi_n\left(x^\varkappa, w_1, \ldots, w_{d+1}\right)
\end{aligned}
\;\right\}
\begin{aligned}
&\varepsilon = 2, \ldots, d+1 \\
&\varkappa = 1, \ldots, n
\end{aligned}
\quad (12)$$

Da $\overset{0}{w_1}$ nicht verschwindet, sind auch die Funktionen $\varphi_{d+2}, \ldots, \varphi_n$ analytisch in der Umgebung von $\overset{0}{x^\varkappa}, w_\lambda$.

Das Problem ist also jetzt eine in der Umgebung von $\overset{0}{x^\varkappa}$ analytische Funktion $s(x^\varkappa)$ zu finden, so dass

$$\left.
\begin{aligned}
\partial_{d+2}\,s &= \varphi_{d+2}\left(x^\varkappa, \partial_1 s, \ldots, \partial_{d+1} s\right) \\
&\;\vdots \\
\partial_n\,s &= \varphi_n\left(x^\varkappa, \partial_1 s, \ldots, \partial_{d+1} s\right)
\end{aligned}
\;\right\}
\varkappa = 1, \ldots, n \quad . \quad . \quad (13)$$

ist, während ausserdem gelten soll

$$\left(\partial_\lambda\,s\right)_{x^\varkappa = \overset{0}{x^\varkappa}} = \overset{0}{w_\lambda}. \quad \ldots \ldots \ldots \quad (14)$$

Für $d = n - 2$ enthält das System (13) nur eine einzige Gleichung. Nach dem Existenztheorem von CAUCHY—KOWALEWSKI gibt es in diesem Falle eine einzige Lösung s, die für $x^n = \underset{0}{x^n}$ in eine beliebige in der Umgebung von $\underset{0}{x^\varkappa}$ analytische Funktion $s(x^1, \ldots, x^{n-1})$ übergeht. Es gibt also unendlich viele Lösungen, die für $x^\varkappa = \underset{0}{x^\varkappa}$ nur den Bedingungen $\partial_1 s = \underset{0}{w_1}, \ldots, \partial_{n-1} s = \underset{0}{w_{n-1}}$ genügen. Da aber $\underset{0}{w_\lambda}$ die Gleichung (12) erfüllt folgt aus (13), dass für jede dieser Lösungen auch $\partial_n s = w_n$ ist in $\underset{0}{x^\varkappa}$. Für $d = n - 2$ ist das Existenztheorem damit bewiesen.

Wir nehmen jetzt an der Beweis sei schon geliefert für eine d-dimensionale vollständige irreduzibele algebraische Mannigfaltigkeit in einer P_{n-2}, deren Grundkörper aus den meromorphen Funktionen in einem Gebiet einer X_{n-1} besteht (vgl. A.S.I[4]) S. 957). Den Induktionsbeweis teilen wir in drei Teile.

a. *Konstruktion der Hilfsfunktion $\underset{0}{s}(x^1, \ldots, x^{n-1})$.*

Es sei $\mathfrak{M}_1$ die projizierende Mannigfaltigkeit von $\mathfrak{M}$ i.b. auf $\overset{n}{e_\lambda}$ (vgl. A.S.I S. 961). Diese wird dargestellt durch die Gleichungen

$$\left. \begin{aligned} w_{d+2} &= \varphi_{d+2}\left(x^\varkappa, w_1, \ldots, w_{d+1}\right) \\ &\;\;\vdots \\ w_{n-1} &= \varphi_{n-1}\left(x^\varkappa, w_1, \ldots, w_{d+1}\right) \end{aligned} \right\} \varkappa = 1, \ldots, n \quad . \quad . \quad (15)$$

Nach A.S.I§5 ist $\mathfrak{M}_1$ sowohl vollständig als irreduzibel, während aus (15) folgt, dass $\mathfrak{M}_1$ ausserdem $(d+1)$-dimensional ist. Man beweist leicht, dass $\mathfrak{M}_1$ für den uns allein noch interessierenden Fall dass $d < n - 2$ ist in $\underset{0}{x^\varkappa}$, $\underset{0}{w_\lambda}$ einen Tangentialraum besitzt[5]) und dass sich $n - 1 - (d + 1)$ kegelartige algebraische Hyperflächen bilden lassen, die $\overset{n}{e_\lambda}$ als Spitze haben, $\mathfrak{M}_1$ enthalten und in $\underset{0}{x^\varkappa}$, $\underset{0}{w_\lambda}$ Tangentialräume besitzen, die linear unabhängig sind. $\mathfrak{M}_1$ ist in einer Umgebung von $\underset{0}{x^\varkappa}$, $\underset{0}{w_\lambda}$ der vollständige Durchschnitt dieser Hyperflächen

$$F_k(x^\varkappa, w_\beta) = F_k^{\beta_1 \ldots \beta_{r_k}} w_{\beta_1} \ldots w_{\beta_{r_k}} = 0; \quad k = d+2, \ldots, n-1 \;\; \beta = 1, \ldots, n-1 \quad (15a)$$

---

[4]) Mit A. S. I bezeichnen wir die erste Mitteilung, Proc. Ned. Akad. v. Wet. **43**, S. 955—963 (1940).

[5]) Wesentlich hierfür ist, dass die durch $\underset{0}{w_\lambda}$ und $\overset{n}{e_\lambda}$ gelegte Verbindungsgerade nur den Punkt $\underset{0}{w_\lambda}$ mit $\mathfrak{M}$ gemeinschaftlich hat.

sodass das System (15a) mit dem System (15) gleichbedeutend ist. Setzt man nun $x^n = \underset{0}{x^n}$ in (15) bzw. (16) so entsteht

$$
\left.
\begin{aligned}
w_{d+2} &= \varphi_{d+2}\left(x^\alpha, \underset{0}{x^n}, w_1, \ldots, w_{d+1}\right) \\
&\;\;\vdots \\
w_{n-1} &= \varphi_{n-1}\left(x^\alpha, \underset{0}{x^n}, w_1, \ldots, w_{d+1}\right)
\end{aligned}
\right\} \quad \alpha = 1, \ldots, n-1 \quad . \quad (16)
$$

bzw.

$$
F_k\left(x^\alpha, \underset{0}{x^n}, w_\beta\right) = 0 \; ; \; \alpha, \beta = 1, \ldots, n-1 \; ; \; k = d+2, \ldots, n-1 \quad . \quad (16a)
$$

und jedes dieser Systeme bestimmt in der P_{n-2}, die zu der X_{n-1} mit der Gleichung $x^n = \underset{0}{x^n}$ gehört, die selbe algebraische Mannigfaltigkeit $\mathfrak{M}_0$, von der man leicht beweist, dass sie vollständig ist (vgl. A. S. I, S. 962) sowie irreduzibel und d-dimensional. Ferner dass in $\underset{0}{x^\alpha}$, w_β ein Tangentialraum existiert, und dass auch die lokale zu $\mathfrak{M}_0$ gehörige algebraische Mannigfaltigkeit in $\underset{0}{x^\alpha}$ irreduzibel ist. Nach der Voraussetzung existiert also eine Funktion $\underset{0}{s}(x^\alpha)$; $\alpha = 1, \ldots, n-1$, die den Gleichungen

$$
\left.
\begin{aligned}
\partial_{d+2}\,\underset{0}{s} &= \varphi_{d+2}\left(x^\alpha, \underset{0}{x^n}, \partial_1\,\underset{0}{s}, \ldots, \partial_{d+1}\,\underset{0}{s}\right) \\
&\;\;\vdots \\
\partial_{n-1}\,\underset{0}{s} &= \varphi_{n-1}\left(x^\alpha, \underset{0}{x^n}, \partial_1\,\underset{0}{s}, \ldots, \partial_{d+1}\,\underset{0}{s}\right)
\end{aligned}
\right\} \quad \alpha = 1, \ldots, n-1 \quad . \quad (17)
$$

oder auch dem mit diesem System gleichwertigen System

$$
F_k\left(x^\alpha, \underset{0}{x^n}, \partial_\beta\,\underset{0}{s}\right) = 0 \; ; \; k = d+2, \ldots, n-1 \quad \alpha, \beta = 1, \ldots, n-1 \quad . \quad (17a)
$$

genügt und für welche in $\underset{0}{\bar{x}^\alpha}$ ausserdem noch gilt

$$
\partial_\beta\,\underset{0}{s} = \underset{0}{w_\beta} \; ; \quad \beta = 1, \ldots, n-1 \quad . \quad . \quad . \quad . \quad . \quad (18)
$$

b. Konstruktion der Funktion $s(x^\varkappa)$.

Nach dem Existenztheorem von CAUCHY—KOWALEWSKI existiert eine einzige Funktion $s(x^\varkappa)$, die der letzten der Gleichungen (13):

$$
\partial_n s = \varphi_n\left(x^\varkappa, \partial_1 s, \ldots, \partial_{d+1} s\right) \quad . \quad . \quad . \quad . \quad . \quad (19)
$$

genügt, während ausserdem gilt

$$
s\left(x^\alpha, \underset{0}{x^n}\right) = \underset{0}{s}(x^\alpha) \; ; \quad \alpha = 1, \ldots, n-1. \quad . \quad . \quad . \quad (20)
$$

Aus (18) folgt für $x^\varkappa = \underset{0}{x^\varkappa}$

$$\partial_\beta s = \underset{0}{w_\beta} \quad ; \quad \beta = 1, \ldots, n-1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

und aus (19) folgt also, dass für $x^\varkappa = \underset{0}{x^\varkappa}$

$$\partial_n s = \varphi_n (\underset{0}{x^\varkappa}, \underset{0}{w_1}, \ldots, \underset{0}{w_{d+1}}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

ist. Da aber $\underset{0}{x^\varkappa}, w_\lambda$ zu $\mathfrak{M}$ gehört und also der Gleichung (12) genügt, folgt in $x^\varkappa = \underset{0}{x^\varkappa}$

$$\partial_n s = \underset{0}{w_n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

sodass die Funktion s der Gleichung (14) genügt.

c. *Beweis dass s auch den ersten $n-d-2$ Gleichungen (13) genügt.*

Der Voraussetzung nach liegt $\overset{n}{e_\lambda}$ nicht in dem Tangentialraum zu $\underset{0}{x^\varkappa}, \underset{0}{w_\lambda}$. Es gibt also mindestens eine Hyperfläche mit einer Gleichung

$$F(x^\varkappa, w_\lambda) = F^{\lambda_1 \ldots \lambda_r} w_{\lambda_1} \ldots w_{\lambda_r} = 0, \quad . \quad . \quad . \quad . \quad (24)$$

mit $F^{\lambda_1 \ldots \lambda_r}$ analytisch in einer Umgebung von $\underset{0}{x^\varkappa}$, die $\mathfrak{M}$ enthält und deren Tangentialraum zu $\underset{0}{x^\varkappa}, w_\lambda$ nicht durch $\overset{n}{e_\lambda}$ geht. Infolgedessen ist

$$\frac{\partial F}{\partial w_n} \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in $\underset{0}{x^\varkappa}, w_\lambda$. Setzt man nun in F für w_n ein $\varphi_n(x^\varkappa, w_\eta)$; $\eta = 1, \ldots, d+1$, so entsteht eine Funktion

$$E(x^\varkappa, w_\beta) \overset{def}{=} F(x^\varkappa, w_\beta, \varphi_n(x^\varkappa, w_\eta)) \ , \ \varkappa = 1, \ldots, n \quad \beta = 1, \ldots, n-1 \left.\begin{array}{l} \\ \eta = 1, \ldots, d+1 \end{array}\right\} \quad (26)$$

die in der Umgebung von $\underset{0}{x^\varkappa}, w_\lambda$ analytisch ist und auf $\mathfrak{M}_1$ verschwindet. Denn ist $w_\beta(x^\varkappa)$ ein beliebiger Punkt von $\mathfrak{M}_1$, so genügt w_β den Gleichungen (15) und daraus folgt, dass $(w_\beta, \varphi_n(x^\varkappa, w_\eta))$ den Gleichungen (12) genügt und also ein Punkt von $\mathfrak{M}$ ist und infolgedessen $F(x^\varkappa, w_\lambda)$ zum Verschwinden bringt.

Das Verschwinden von E auf $\mathfrak{M}_1$ bringt mit sich, dass es eine Gleichung gibt von der Form

$$E(x^\alpha, w_\beta) = J^k(x^\varkappa, w_\beta) F_k(x^\varkappa, w_\beta) \ ; \ k = d+2, \ldots, n-1, \left.\begin{array}{l} \\ \beta = 1, \ldots, n-1 \\ \varkappa = 1, \ldots, n \end{array}\right\} \quad . \quad . \quad (27)$$

wo die rechts auftretenden Funktionen J^k (die nicht notwendig rational in den w_β zu sein brauchen) in der Umgebung von $\overset{0}{x^\nu}$, $\overset{0}{w_\beta}$ analytisch in x^ν, w_β sind [6]). Setzt man nun $w_\beta = \partial_\beta s$ in (27), so entsteht infolge (19)

$$F(x^\varkappa, \partial_\lambda s) = E(x^\varkappa, \partial_\beta s) = J^k(x^\varkappa, \partial_\beta s) F_k(x^\varkappa, \partial_\beta s); \left.\begin{array}{l} k = d+2, \ldots, n-1 \\ \beta = 1, \ldots, n-1 \\ \varkappa, \lambda = 1, \ldots, n \end{array}\right\} \quad . \quad . \quad (28)$$

wo die $J^k(x^\nu, \partial_\beta s)$ jetzt in der Umgebung von $\overset{0}{x^\nu}$ analytisch sind in $x^\varkappa$. Es verschwinden nun sowohl $F(x^\nu, w_\lambda)$ als $F_k(x^\nu, w_\lambda)$, (w_n tritt in den F_k nicht auf), auf $\mathfrak{M}$. Da $\mathfrak{M}$ vollständig ist, verschwinden daselbst also auch die Polynome

$$K_k(x^\varkappa, w_\lambda) \overset{def}{=} D(F_k; F)^{\mu\lambda_1 \ldots \lambda_r}{}_k{}^{\varkappa_1 \ldots \varkappa_r} w_\mu w_{\lambda_1} \ldots w_{\varkappa_r} \left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\} \quad . \quad . \quad . \quad (29)$$
$$k = d+2, \ldots, n-1$$

Für diese Polynome gelten also genau wie für F Gleichungen von der Form

$$K_k(x^\varkappa, \partial_\lambda s) = L_k^l(x^\varkappa, \partial_\beta s) F_l(x^\varkappa, \partial_\beta s), \quad k, l = d+2, \ldots, n-1 \quad (30)$$

wo die $L_k^l(x^\varkappa, \partial_\beta s)$ analytisch in x^ν sind in der Umgebung von $\overset{0}{x^\varkappa}$. Schreiben wir $F_k(x^\nu, \partial_\beta s) \overset{def}{=} \theta_k(x^\nu)$, so ist (30) gleichbedeutend mit den Kongruenzen

$$K_k(x^\varkappa, \partial_\lambda s) = D(F_k; F)^{\mu\lambda_1 \ldots \lambda_r}{}_k{}^{\varkappa_1 \ldots \varkappa_r}(\partial_\mu s)(\partial_{\lambda_1} s) \ldots (\partial_{\varkappa_r} s) \equiv 0 \, mod\,(\theta_l) \left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\}\!\! \quad (31)$$
$$k, l = d+2, \ldots, n-1$$

Auf diese Kongruenzen wenden wir jetzt die allgemeine Formel

$$D(X_a; X_b)^{\mu\lambda_1 \ldots \lambda_{a+b}} w_\mu w_{\lambda_1} \ldots w_{\lambda_{a+b}} = 2(a+1)(b+1) X_a^{\mu\lambda_1 \ldots \lambda_a} \left.\vphantom{\begin{array}{c}a\\b\\c\\d\end{array}}\right\}$$
$$X^{\mu\nu_1 \ldots \nu_b}(\partial_{[\mu} w_{\lambda]}) w_{\lambda_1} \ldots w_{\nu_b} + \left|\vphantom{\begin{array}{c}a\\b\end{array}}\right.$$
$$+ (a+1) X_a^{\mu\lambda_1 \ldots \lambda_a} w_{\lambda_1} \ldots w_{\lambda_a} \partial_\mu(X_b^{\lambda\nu_1 \ldots \nu_b} w_\lambda w_{\nu_1} \ldots w_{\nu_b}) - \left|\vphantom{\begin{array}{c}a\\b\end{array}}\right.$$
$$- (b+1) X_b^{\mu\lambda_1 \ldots \lambda_b} w_{\lambda_1} \ldots w_{\lambda_b} \partial_\mu(X_a^{\lambda\nu_1 \ldots \nu_a} w_\lambda w_{\nu_1} \ldots w_{\nu_a}) \left.\vphantom{\begin{array}{c}a\\b\\c\\d\end{array}}\right\} \quad (32)$$

an, die eine Verallgemeinerung ist der Formel (10) A. S. I, S. 958 und in diese Formel übergetzt für den dort berücksichtigten Fall, dass die

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[6]) Vergl. für den Beweis dieser Tatsache E. KÄHLER, Einführung in die Theorie der Systeme von Differentialgleichungen, S. 12.

eingeklammerten Ausdrücke in den beiden letzten Termen verschwinden. Da $\dfrac{\partial F_k}{\partial w_n}$ verschwindet, ergibt sich dann

$$\left\{\frac{\partial F_k}{\partial w_\beta}(x^\varkappa, \partial_\lambda s)\right\}\partial_\beta F(x^\varkappa, \partial_\lambda s) - \left\{\frac{\partial F}{\partial w^\mu}(x^\varkappa, \partial_\lambda s)\right\}\partial_\mu \theta_k \equiv 0 \; mod \, (\theta_l) \left.\begin{array}{c} \\ \\ k, l = d+2, \ldots, n-1 \\ \beta = 1, \ldots, n-1 \\ \varkappa, \lambda, \mu = 1, \ldots, n \end{array}\right\} . \; (33)$$

Da aber infolge (28)

$$\partial_\beta F(x^\varkappa, \partial_\lambda s) \equiv 0 \; mod \, (\theta_k, \partial_\beta \theta_k) \left.\begin{array}{c} \\ \beta = 1, \ldots, n-1 \\ \varkappa, \lambda = 1, \ldots, n \\ k = d+2, \ldots, n-1 \end{array}\right\} \; \ldots \ldots \; (34)$$

folgt

$$\left\{\frac{\partial F}{\partial w_n}(x^\varkappa, \partial_\lambda s)\right\}\partial_n \theta_k(x^\varkappa) \equiv 0 \; mod \, (\theta_l, \partial_\beta \theta_l) \left.\begin{array}{c} \\ \beta = 1, \ldots, n-1 \\ \varkappa, \lambda = 1, \ldots, n \\ k, l = d+2, \ldots, n-1 \end{array}\right\} \; \ldots \; (35)$$

wobei zu bemerken ist, dass beim Ausschreiben des rechten Gliedes in $\theta_{d+2}, \ldots, \partial_\beta \theta_{n-1}$ alle Koeffizienten analytisch sind in der Umgebung von $x^\varkappa_0$. Da in diesem Punkte $\dfrac{\partial F}{\partial w_n}_0 (x^\varkappa, \partial_\lambda s)$ übergeht in $\dfrac{\partial F}{\partial w_n}_0 (x^\varkappa, \overset{0}{w_\lambda})$ und dieser Ausdruck infolge (25) nicht verschwindet, kann man durch $\dfrac{\partial F}{\partial w_n}$ dividieren und es entsteht eine Gleichung von der Form

$$\partial_n \theta_k(x^\varkappa) = \varphi^l_k(x^\varkappa)\, \theta_l(x^\varkappa) + \varphi^{l,\beta}_k(x^\varkappa)\, \partial_\beta \theta_l(x^\varkappa) \left.\begin{array}{c} \\ k, l = d+2, \ldots, n-1 \\ \beta = 1, \ldots, n-1 \\ \varkappa = 1, \ldots, n \end{array}\right\} \; \ldots \; (36)$$

mit in der Umgebung von $x^\varkappa_0$ analytischen Koeffizienten im rechten Gliede. Nun ist ausserdem

$$\theta_k(\overset{0}{x^\alpha}, x^n) = F_k(\overset{0}{x^\alpha}, x^n, \partial_\beta s) = 0 \left.\begin{array}{c} \\ \alpha, \beta = 1, \ldots, n-1 \\ k = d+2, \ldots, n-1 \end{array}\right\} \; \ldots \ldots \; (37)$$

infolge (17a) und (20). Nach dem Existenztheorem von CAUCHY—
KOWALEWSKI hat das System (36) mit den Anfangsbedingungen (37) eine
einzige Lösung. Da aber $\theta_k(x^\nu) = 0$ offenbar eine Lösung ist, ist dies
auch die einzige Lösung und damit ist gezeigt, dass $\partial_\lambda s$ auch die
Gleichung (15a) und somit auch die mit dieser gleichwertige Gleichung
(15) befriedigt, woraus hervorgeht dass s nicht nur der letzten der
Gleichungen (13) sondern sämtlichen Gleichungen (13) genügt w.z.b.w.

8. *Schluss.*
Wir kehren jetzt zurück zu der in A. S. I § 2 gestellten Aufgabe die
Lösungen zu bestimmen der Systeme

$$\underset{i}{P}{}^{\mu\lambda_1\ldots\lambda}a_i\,(\partial_\mu s)\,(\partial_{\lambda_1} s)\ldots(\partial_{\lambda_{a_i}} s) = 0 \; ; \; i = 1,\ldots, N. \quad . \quad . \quad (38)$$

Zunächst bestimme man nach der in A. S. I § 3 angegebenen Methode
die grösste vollständige Mannigfaltigkeit $\mathfrak{M}_\nu$. die in der Mannigfaltigkeit
$\mathfrak{M}$, definiert durch die Gleichungen

$$\underset{i}{P}{}^{\mu\lambda_1\ldots\lambda}a_i\,w_\mu\,w_{\lambda_1}\ldots w_{\lambda_{a_i}} = 0 \; ; \; i = 1,\ldots, N \quad . \quad . \quad . \quad (39)$$

enthalten ist. Sodann zerlege man $\mathfrak{M}_\nu$ in irreduzibele Mannigfaltigkeiten
$\mathfrak{M}_{\nu 1},\ldots, \mathfrak{M}_{\nu r}$, die alle vollständig sind. Sind dann z.B.

$$\underset{s}{Q}{}^{\mu\lambda_1\ldots\lambda}a_s\,w_\mu\,w_{\lambda_1}\ldots w_{\lambda_{a_s}} = 0 \; ; \; s = 1,\ldots, M \quad . \quad . \quad . \quad (40)$$

die Gleichungen von $\mathfrak{M}_{\nu 1}$, so bestimme man in einem Punkt $\underset{0}{x^\varkappa}$ eine
Lösung $\underset{0}{w_\lambda}$ dieser Gleichungen, so dass auch die lokale zu $\mathfrak{M}_{\nu 1}$ gehörige
Mannigfaltigkeit irreduzibel ist und $\mathfrak{M}_{\nu 1}$ in $\underset{0}{x^\varkappa}, \underset{0}{w_\lambda}$ einen Tangentialraum
hat. Es lässt sich dann eine Funktion s bilden, so dass $\partial_\lambda s$ (40) also
auch (38) befriedigt und für welche ausserdem in $\underset{0}{x^\varkappa}$ gilt $\partial_\lambda s = \underset{0}{w_\lambda}$.
In dieser Weise können alle Lösungen von (38) erhalten werden.
Weniger genau lässt sich das Existenztheorem auch folgendermassen
formulieren:

*Ist $\underset{0}{w_\lambda}$ in $\underset{0}{x^\varkappa}$ ein zu einer vollständigen (nicht notwendig irreduzibelen)
algebraischen Mannigfaltigkeit gehöriger Vektor nicht besonderer Lage,
so lässt sich $\underset{0}{w_\lambda}$ zu einem zu $\mathfrak{M}$ gehörigen Gradientfeld fortsetzen*

oder auch

Die zu einer vollständigen algebraischen Mannigfaltigkeit gehörigen

Vektoren nicht besonderer Lage lassen sich zu Gradientfeldern anein-
anderreihen.

Es sei noch bemerkt, dass das Existenztheorem folgende leicht beweis-
bare Umkehrung zulässt.

Lässt sich in jedem beliebigen Punkt $x^\varkappa_0$ eines bestimmten Gebietes
der X_n jeder zu einer algebraischen Mannigfaltigkeit $\mathfrak{M}$ gehörige Vektor
w_λ eines bestimmten Gebietes des w_λ-Raumes in diesem Punkt in einer
Umgebung von $x^\varkappa_0$ zu einem zu $\mathfrak{M}$ gehörigen Gradientfeld fortsetzen,
so ist $\mathfrak{M}$ vollständig.

Chemistry. — *Thixotropy of suspensions in apolar liquids.* (Preliminary Communication.) By H. R. KRUYT and F. G. VAN SELMS.

FREUNDLICH and RÖDER [1]) have made the interesting observation that suspensions of rice starch or quartz, which in water give systems that may be considered Newtonian liquids, occasion spontaneous thixotropy (so-called "false body") in CCl_4, cyclohexane respectively. One of us [2]) has attached theoretical consequences to this observation which affect not only the theory of thixotropy, but also the theory of gelatination generally.

In order to test and extend this course of thought an extensive investigation is being made, following a method different from RÖDER's, which has revealed facts that for the present throw a different light on the observations mentioned. It appears that thixotropy does, indeed, occur when starch is suspended in air dry condition (that is with a moisture percentage of ca. 12 %) in CCl_4, but that the phenomenon is *practically absent* when the starch is first dried in a vacuum dessiccator over chloride of calcium. In the same way suspensions of finely divided quartz *do not show thixotropy* in CCl_4 when this material has first been dehydrated during $1\frac{1}{2}$ hours at ca. 360°.

Hence the explanation given so-far for the remarkable phenomenon of suspensions in an apolar medium is in need of revision. The presence of water in the phase boundary of, for instance quartz — CCl_4, apparently plays an all-important part. One is even inclined to think that it is not the apolarity of the organic liquid that causes the phenomenon but its "unmixing" with water.

Although the investigations are as yet only in the initial stage, and we hope to return to this problem more extensively at a later date, we considered these facts of such importance for the theory of thixotropy and gelatination, as to justify the immediate publication of this experience.

VAN 'T HOFF-*Laboratorium.*

Utrecht, October 1940.

[1]) H. FREUNDLICH and H. L. RÖDER, Trans. Faraday Soc. **34**, 308 (1938).
[2]) H. R. KRUYT, Chimie et Industrie **42**, 587 (1939).

Mathematics. — *Un problème de la théorie des réseaux plans.* Par
P. Hazebroek. (Communicated by Prof. W. van der Woude.)

1. Deux familles de courbes dans le plan constituent dans une région R un réseau plan, quand: $1°$ par chaque point de R passe une courbe de chaque famille, de telle sorte que ces deux courbes ne soient pas tangentes, $2°$ deux courbes appartenant à la même famille n'ont pas de point en commun.

Un réseau est exprimé analytiquement par trois équations $x = x(u, v)$, dans lesquelles le seul symbole x représente les trois coordonnées projectives planes d'un point x par rapport à un repère fixe. On obtient les deux familles de courbes en ne variant qu'*un* des paramètres u, v.

Par deux droites conjuguées en un point x d'un réseau nous entendons deux droites passant par x et harmoniques aux tangentes $x x_u$, $x x_v$. Par un réseau conjugué d'un réseau donné N nous entendons un réseau, dont toutes les paires de tangentes sont droites conjuguées par rapport à N.

Le sujet de notre article est la solution du problème suivant:

Y-a-t'il des réseaux, qui admettent une infinité de réseaux conjugués, dont les deux familles soient constituées exclusivement de droites?

Nous supposons qu'aucune des deux familles du réseau cherché ne soit formée uniquement de droites.

2. En traitant ce problème nous ferons usage de la méthode des repères mobiles de Cartan, dont le lecteur est supposé au courant.

A chaque point de notre réseau nous associons un repère local aux sommets a_0, a_1, a_2, dont les coordonnées par rapport au repère fixe satisfont la relation

$$[a_0, a_1, a_2] = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Les coordonnées fixes d'un point sont des combinaisons linéaires des coordonnées de a_0, a_1 et a_2, de sorte qu'elles puissent être exprimées par l'équation

$$z = z_0\, a_0 + z_1\, a_1 + z_2\, a_2,$$

où z_0, z_1, z_2 sont les coordonnées locales de z.

De la même façon les coordonnées de da_0, da_1, da_2 sont exprimées par les équations

$$\left. \begin{aligned} d a_0 &= \omega_{00}\, a_0 + \omega_{01}\, a_1 + \omega_{02}\, a_2 \\ d a_1 &= \omega_{10}\, a_0 + \omega_{11}\, a_1 + \omega_{12}\, a_2 \\ d a_2 &= \omega_{20}\, a_0 + \omega_{21}\, a_1 + \omega_{22}\, a_2, \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (2)$$

dans lesquelles les ω_{ij} sont des expressions de PFAFF. En vertu de (1) nous avons

$$\omega_{00} + \omega_{11} + \omega_{22} = 0 . . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Les covariants bilinéaires des équations (2) doivent être 0, d'où il suit que

$$\omega'_{ij} = \sum_k [\omega_{ik}, \omega_{kj}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Le repère général dépend de 8 paramètres. Nous attacherons le repère au réseau de façon covariante, de sorte qu'il ne dépende que des deux paramètres essentiels u, v du réseau, les six paramètres inessentiels doivent être éliminés. Une différentiation qui ne fait varier que les paramètres inessentiels sera désignée par le symbole δ; en ce cas nous écrirons e_{ij} au lieu de ω_{ij}.

Nous considérons un point $x(u_0, v_0)$ de notre réseau et faisons coïncider a_0 et x, tandis que $a_0 a_1$ et $a_0 a_2$ sont des tangentes respectivement aux courbes $v = $ const. et $u = $ const. passant par x. En outre nous faisons coïncider a_1 et a_2 avec les transformés de LAPLACE correspondant respectivement aux directions de u et de v. Les points *géométriques* a_0, a_1 et a_2 sont alors définis complètement et déterminés analytiquement à un facteur près, avec la restriction résultant de l'équation (1). Il n'y a donc encore à éliminer que deux paramètres inessentiels. Nous avons encore

$$\delta a_i = e_{ii} a_i . \quad (i = 1, 2, 3)$$

Tous les autres e_{ij} sont 0.

Par suite du choix de notre repère local nous avons $\omega_{01} = f(u, v) du$, $\omega_{02} = g(u, v) dv$, nous écrirons dorénavant $\omega_{01} = \omega_1$, $\omega_{02} = \omega_2$. Tous les ω_{ij}, pour lesquels $e_{ij} = 0$, sont des formes linéaires de du et dv, par conséquent aussi de ω_1 et ω_2. En particulier à cause du choix de a_1 et a_2 on a $\omega_{12} = b_1 \omega_1$, $\omega_{21} = b_2 \omega_2$. Puisque nous supposons qu'aucune des deux familles du réseau soit constituée uniquement de droites, b_1 et b_2 sont $\neq 0$.

De (4) il suit, en rappelant que $[\omega_{21}, \omega_2]$ et $[\omega_{12}, \omega_1]$ sont 0

$$\omega'_1 = [\omega_{00} - \omega_{11}, \omega_1], \quad \omega'_2 = [\omega_{00} - \omega_{22}, \omega_2]. \quad . \quad . \quad . \quad (5)$$

Donc

$$\delta \omega_1 = (e_{00} - e_{11}) \omega_1, \quad \delta \omega_2 = (e_{00} - e_{22}) \omega_2.$$

De même

$$\left. \begin{aligned} \omega'_{12} &= [\omega_{10}, \omega_2] + [\omega_{11} - \omega_{22}, \omega_{12}] \\ \omega'_{21} &= [\omega_{20}, \omega_1] + [\omega_{22} - \omega_{11}, \omega_{21}], \end{aligned} \right\} \quad . \quad . \quad . \quad (6)$$

d'où il s'ensuit, en vertu de la formule $(k\omega)' = k\omega' + [dk, \omega]$,

$$\delta \omega_{12} = b_1 (e_{11} - e_{22}) \omega_1 = \omega_1 \delta b_1 + b_1 \delta \omega_1$$
$$\delta \omega_{21} = b_2 (e_{22} - e_{11}) \omega_2 = \omega_2 \delta b_2 + b_2 \delta \omega_2$$

de sorte que, tenant compte de (3)

$$\delta b_1 = 3\, e_{11}\, b_1, \quad \delta b_2 = 3\, e_{22}\, b_2.$$

Il est donc permis de déterminer les deux paramètres inessentiels de telle façon que $b_1 = 1$ et $b_2 = 1$. Par là e_{11} et e_{22} deviennent 0 et notre repère est maintenant fixé aussi analytiquement.

De (4), (5) et (6) il découle:

$$[3\,\omega_{11},\ \omega_1] + [\omega_{10},\quad \omega_2] = 0$$
$$[\omega_{20},\quad \omega_1] + [3\omega_{22},\ \omega_2] = 0.$$

En vertu de ces relations nous posons

$$\left. \begin{aligned} \omega_{00} &= -(b + \beta)\,\omega_1 - (c + \gamma)\,\omega_2 \\ \omega_{11} &= b\,\omega_1 + \gamma\,\omega_2, \quad \omega_{10} = 3\,\gamma\,\omega_1 + \mu\,\omega_2, \\ \omega_{22} &= \beta\,\omega_1 + c\,\omega_2, \quad \omega_{20} = \nu\,\omega_1 + 3\,\beta\,\omega_2, \end{aligned} \right\} \quad \ldots \ldots \quad (7)$$

où $b, c, \beta, \gamma, \mu, \nu$ sont des invariants du réseau. Les équations de notre réseau sont donc

$$\left. \begin{aligned} d\,a_0 &= \{-(b + \beta)\,\omega_1 - (c + \gamma)\,\omega_2\}\,a_0 + \omega_1\,a_1 + \omega_2\,a_2 \\ d\,a_1 &= (3\,\gamma\,\omega_1 + \mu\,\omega_2)\,a_0 + (b\,\omega_1 + \gamma\,\omega_2)\,a_1 + \omega_1\,a_2 \\ d\,a_2 &= (\nu\,\omega_1 + 3\,\beta\,\omega_2)\,a_0 + \omega_2\,a_1 + (\beta\,\omega_1 + c\,\omega_2)\,a_2. \end{aligned} \right\} \quad \ldots \ldots \quad (8)$$

$$\left. \begin{aligned} \omega_1' &= (2\,\gamma + c)\,[\omega_1, \omega_2] \\ \omega_2' &= -(2\,\beta + b)\,[\omega_1, \omega_2]. \end{aligned} \right\} \quad \ldots \ldots \ldots \quad (9)$$

Pour une fonction arbitraire F nous posons

$$dF = F_1\,\omega_1 + F_2\,\omega_2.$$

On peut alors déduire des conditions d'intégrabilité (4) les relations suivantes

$$\left. \begin{aligned} \gamma_1 - b_2 &= 2\,\beta\gamma - b\gamma - bc - \mu + 1 \\ \beta_2 - c_1 &= 2\,\beta\gamma - c\beta - bc - \nu + 1, \end{aligned} \right\} \quad \ldots \ldots \ldots \quad (10)$$

$$\left. \begin{aligned} \mu_1 - 3\,\gamma_2 &= 3\,\mu\,(\beta + b) - 6\,\gamma\,(2\,\gamma + c) + 3\,\beta \\ \nu_2 - 3\,\beta_1 &= 3\,\nu\,(\gamma + c) - 6\,\beta\,(2\,\beta + b) + 3\,\gamma. \end{aligned} \right\} \quad \ldots \ldots \quad (11)$$

Nous examinons encore dans quelles conditions le réseau est asymptotique. Si nous entendons par d_1 et d_2 des symboles de différentiation dans les directions de u et de v, de manière que $d_1 F = F_1\omega_1$, $d_2 F = F_2\omega_2$, il est facile à trouver que le rapport anharmonique des points $a_0, a_1, a_0 + d_1a_0$, $a_1 + d_2a_1$ est égal à $\mu\omega_1\omega_2$. De même le rapport anharmonique des points $a_0, a_2, a_0 + d_2a_0, a_2 + d_1a_2$ est égal à $\nu\omega_1\omega_2$. Si ces deux rapports anharmoniques sont égaux entre eux, donc si $\mu = \nu$, le réseau est asymptotique [1]).

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[1]) Selon la définition plus usuelle un réseau est asymptotique, s'il peut être considéré comme la projection des asymptotiques d'une surface. Pour faciliter les calculs nous nous sommes servis de la propriété employée dans le texte.

3.   Soit $\omega_2{}^2 = k^2 \omega_1{}^2$ $(k = k(u, v))$ l'équation differentielle d'un réseau conjugué de notre réseau. La tangente à la courbe $\omega_2 = k\omega_1$ en a_0 est $kz_1 - z_2 = 0$. La courbe $\omega_2 = k\omega_1$ sera une droite, si l'équation $kz_1 - z_2 = 0$ ne change pas en différentiant dans la direction $\omega_2 = k\omega_1$.

Pour qu'une droite l

$$h_0 z_0 + h_1 z_1 + h_2 z_2 = 0$$

reste fixe en différentiant dans une certaine direction, il faut que les coefficients de l'équation

$$(d h_0 - \Sigma \omega_{0i} h_i) z_0 + (d h_1 - \Sigma \omega_{1i} h_i) z_1 + (d h_2 - \Sigma \omega_{2i} h_i) z_2 = 0$$

soient proportionnels aux coefficients de l'équation de l. En se servant de cela pour la droite $kz_1 - z_2 = 0$, nous trouvons par différentiation dans la direction $\omega_2 = k\omega_1$ que k doit satisfaire à la condition

$$k_1 + k k_2 + 1 + k (\beta - b) - k^2 (\gamma - c) - k^3 = 0. \quad . \quad . \quad . \quad (12)$$

Pour trouver la condition dans laquelle la courbe $\omega_2 = -k\omega_1$ est une droite, nous remplaçons k par $-k$ dans l'équation (12). Les conditions pour que ces deux équations existent à la fois, sont donc

$$k_1 = -k (\beta - b) + k^3, \quad k k_2 = -1 + k^2 (\gamma - c). \quad . \quad . \quad . \quad (13)$$

Le fait que $dF = F_1\omega_1 + F_2\omega_2$ est une différentielle exacte, est exprimé en posant zéro le covariant bilinéaire; cela nous conduit en vertu de (9) à la relation

$$F_{12} - (2\gamma + c) F_1 = F_{21} - (2\beta + b) F_2. \quad . \quad . \quad . \quad (14)$$

En faisant usage de cette formule on déduit de (13), à l'aide de (10), l'équation pour k:

$$3 c k^4 - (\mu + \nu - 6) k^2 + 3 b = 0. \quad . \quad . \quad . \quad . \quad (15)$$

Pour qu'il y ait une infinité de réseaux conjugués doublement réglés, il faut que les coefficients de l'équation (15) soient 0, donc

$$b = 0, \quad c = 0, \quad \mu + \nu = 6.$$

Nous posons

$$\mu = 3 (1 + \vartheta), \quad \nu = 3 (1 - \vartheta)$$

et commençons par supposer que $\vartheta \neq 0$, donc que le réseau n'est pas asymptotique.

4.   Le cas $\vartheta \neq 0$. Les équations (9), (10) et (11) deviennent alors

$$\omega_1' = 2\gamma [\omega_1, \omega_2], \quad \omega_2' = -2\beta [\omega_1, \omega_2], \quad . \quad . \quad . \quad (16)$$

$$\gamma_1 = 2\beta\gamma - 3\vartheta - 2, \quad \beta_2 = 2\beta\gamma + 3\vartheta - 2, \quad . \quad . \quad . \quad (17)$$

$$\vartheta_1 - \gamma_2 = 3\beta\vartheta - 4\gamma^2 + 4\beta, \quad \vartheta_2 + \beta_1 = 3\gamma\vartheta + 4\beta^2 - 4\gamma. \quad . \quad (18)$$

En vertu de ces deux dernières équations nous introduisons deux nouvelles fonctions ξ et η de telle sorte que

$$\gamma_2 = 4\,\gamma^2 - 4\,\beta + 3\,\xi\,\vartheta, \quad \beta_1 = 4\,\beta^2 - 4\,\gamma - 3\,\eta\,\vartheta, \quad . \quad . \quad . \quad (19)$$

$$\vartheta_1 = 3\,\vartheta\,(\beta + \xi), \quad \vartheta_2 = 3\,\vartheta\,(\gamma + \eta). \quad . \quad . \quad . \quad . \quad (20)$$

Les covariants bilinéaires de $\gamma_1\omega_1 + \gamma_2\omega_2$ et $\beta_1\omega_1 + \beta_2\omega_2$ étant 0, on obtient

$$\left.\begin{aligned} \xi_1 &= \beta\,\xi - 3\,\xi^2 + 9\,\gamma - 7\,\eta \\ \eta_2 &= \gamma\,\eta - 3\,\eta^2 + 9\,\beta - 7\,\xi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

De même pour le covariant bilinéaire de $\vartheta_1\omega_1 + \vartheta_2\omega_2$

$$\xi_2 - \eta_1 = 2\,\gamma\,\xi - 2\,\beta\,\eta - 6\,\vartheta.$$

Nous pouvons donc introduire une fonction φ, de manière que

$$\left.\begin{aligned} \xi_2 &= 2\,\gamma\,\xi - 3\,\vartheta + 7\,\varphi + 5 \\ \eta_1 &= 2\,\beta\,\eta + 3\,\vartheta + 7\,\varphi + 5. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (22)$$

De même façon que ci-dessus on trouve

$$\left.\begin{aligned} \varphi_1 &= 3\,\varphi\,(\beta - 2\,\xi) + 9\,\xi\,\vartheta + 3\,\xi - 12\,\beta + 3\,\eta\,\gamma + 3\,\eta^2 \\ \varphi_2 &= 3\,\varphi\,(\gamma - 2\,\eta) - 9\,\eta\,\vartheta + 3\,\eta - 12\,\gamma + 3\,\xi\,\beta + 3\,\xi^2 \end{aligned}\right\} \quad . \quad (23)$$

avec la condition d'intégrabilité

$$\varphi + \xi\,\eta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

En différentiant et faisant usage des relations (17)—(24) on déduit les nouvelles relations

$$\left.\begin{aligned} \xi^2\,\eta - 2\,\xi + 3\,\beta + \eta^2 - 3\,\eta\,\gamma - 3\,\xi\,\vartheta &= 0 \\ \xi\,\eta^2 - 2\,\eta + 3\,\gamma + \xi^2 - 3\,\xi\,\beta + 3\,\eta\,\vartheta &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

Différentiant la première de ces équations dans la direction de $\omega_2 = 0$ et la seconde dans la direction de $\omega_1 = 0$ on trouve ensuite

$$\left.\begin{aligned} -3\,\gamma\,\vartheta - \xi^3\,\eta + 3\,\gamma\,\xi\,\eta - 3\,\xi\,\eta^2 + 3\,\xi\,\beta - 6\,\gamma + 4\,\eta &= 0 \\ 3\,\beta\,\vartheta - \xi\,\eta^3 + 3\,\beta\,\xi\,\eta - 3\,\xi^2\,\eta + 3\,\eta\,\gamma - 6\,\beta + 4\,\xi &= 0. \end{aligned}\right\} \quad . \quad (26)$$

Résolvant β et γ de (25) on trouve

$$3\,(\xi\,\eta - 1)\,\beta = (\eta^2 + 2\,\xi)\,(\xi\,\eta - 1) + 3\,(\eta^2 - \xi)\,\vartheta$$

$$3\,(\xi\,\eta - 1)\,\gamma = (\xi^2 + 2\,\eta)\,(\xi\,\eta - 1) - 3\,(\xi^2 - \eta)\,\vartheta.$$

Si $\xi\,\eta \neq 1$ on peut substituer β et γ dans (26) et on obtient alors

$$(\xi^2 - \eta)\,(\xi\,\eta - 1 - \tfrac{3}{4}\,\vartheta) = 0$$

$$(\xi - \eta^2)\,(\xi\,\eta - 1 + \tfrac{3}{4}\,\vartheta) = 0.$$

On peut alors distinguer, en supposant toujours que $\vartheta \neq 0$, les cas suivants:

$$1° \quad \xi\eta = 1, \quad \xi = \varepsilon, \quad \eta = \varepsilon^2 \ (\varepsilon^3 = 1).$$
$$2° \quad \xi = \eta = 0.$$
$$3° \quad \xi = \eta^2, \quad \vartheta = \tfrac{1}{3}(\xi\eta - 1).$$

En comparant tous ces cas aux équations précédentes on trouve qu'ils mènent tous au cas $\vartheta = 0$. Les réductions sont tellement simples que nous les avons supprimées.

Nous trouvons donc comme premier résultat que *les réseaux cherchés sont, s'ils existent, asymptotiques.*

5. Le cas $\vartheta = 0$. Les équations (16)—(18) deviennent maintenant

$$\omega_1' = 2\gamma[\omega_1, \omega_2], \quad \omega_2' = -2\beta[\omega_1, \omega_2]; \quad \dots \quad (27)$$
$$\beta_2 = \gamma_1 = 2\beta\gamma - 2; \quad \dots \quad (28)$$
$$\beta_1 = 4\beta^2 - 4\gamma, \quad \gamma_2 = 4\gamma^2 - 4\beta. \quad \dots \quad (29)$$

Il se trouve alors que toutes les conditions d'intégrabilité sont satisfaites.

Afin de déterminer les réseaux pour lesquels ces relations sont valables, nous posons

$$\omega_1 = f\,du, \quad \omega_2 = g\,dv.$$

On a maintenant pour une fonction arbitraire F

$$F_u = f F_1, \quad F_v = g F_2. \quad \dots \quad (30)$$

En outre il suit de (27)

$$\beta = -\frac{g_u}{2fg}, \quad \gamma = -\frac{f_v}{2fg}.$$

De $\beta_2 = \gamma_1$ et (30) il découle

$$(\log f/g)_{uv} = 0,$$

équation qui a la solution générale

$$\frac{f}{g} = \frac{U}{V}$$

où U et V sont des fonctions arbitraires respectivement de u et v. Par une transformation $u = u(\bar{u})$, $v = v(\bar{v})$ ω_1 et ω_2 se changent en $\omega_1 = \bar{f}d\bar{u}$, $\omega_2 = \bar{g}d\bar{v}$, $\bar{f}$ et $\bar{g}$ se transformant en $\bar{f} = f\dfrac{du}{d\bar{u}}$, $\bar{g} = g\dfrac{dv}{d\bar{v}}$. On peut donc transformer les paramètres de telle manière que f devienne égal à g. Les paramètres sont alors déterminés à des constantes additives près.

Nous supprimons maintenant les barres et posons

$$\omega_1 = \frac{du}{\psi}, \quad \omega_2 = \frac{dv}{\psi}.$$

Alors

$$\beta = \tfrac{1}{2}\,\psi_u, \quad \gamma = \tfrac{1}{2}\,\psi_v$$

et les équations (28) et (29) conduisent aux équations suivantes pour ψ

$$\psi\,\psi_{uu} = 2\,\psi_u^2 - 4\,\psi_v,$$
$$\psi\,\psi_{vv} = 2\,\psi_v^2 - 4\,\psi_u,$$
$$\psi\,\psi_{uv} = \psi_u\,\psi_v - 4.$$

Les trois solutions indépendantes de ces équations sont

$$\psi = 2\,(\varepsilon\,\acute{u} + \varepsilon^2\,v), \quad \varepsilon^3 = 1.$$

Considérons d'abord la solution $\psi = 2(u + v)$. Les invariants du réseau sont alors

$$\beta = \gamma = 1, \quad \mu = \nu = 3,$$

et les équations (8) deviennent

$$d a_0 = -\left\{ \frac{du}{2\,(u+v)} + \frac{dv}{2\,(u+v)} \right\} a_0 + \frac{du}{2\,(u+v)}\, a_1 + \frac{dv}{2\,(u+v)}\, a_2,$$

$$d a_1 = 3\left\{ \frac{du}{2\,(u+v)} + \frac{dv}{2\,(u+v)} \right\} a_0 + \frac{dv}{2\,(u+v)}\, a_1 + \frac{du}{2\,(u+v)}\, a_2,$$

$$d a_0 = 3\left\{ \frac{du}{2\,(u+v)} + \frac{dv}{2\,(u+v)} \right\} a_0 + \frac{dv}{2\,(u+v)}\, a_1 + \frac{du}{2\,(u+v)}\, a_2.$$

Il se trouve que les deux autres solutions pour ψ mènent à ces mêmes équations si l'on remplace a_1 et a_2 par εa_2 et $\varepsilon^2 a_2$ et si l'on introduit εu et $\varepsilon^2 v$ comme nouvelles variables.

En éliminant a_1 et a_2 on trouve les équations différentielles suivantes pour le réseau

$$a_{0uu} = \frac{1}{(u+v)^2}\, a_0 - \frac{3}{2\,(u+v)}\, a_{0u} + \frac{1}{2\,(u+v)}\, a_{0v},$$

$$a_{0vv} = \frac{1}{(u+v)^2}\, a_0 + \frac{1}{2\,(u+v)}\, a_{0u} - \frac{3}{2\,(u+v)}\, a_{0v},$$

$$a_{0uv} = \frac{1}{(u+v)^2}\, a_0 - \frac{1}{2\,(u+v)}\, a_{0u} - \frac{1}{2\,(u+v)}\, a_{0v}.$$

6.   Afin de résoudre ces équations nous posons $a_0 = x/(u+v)$ par quoi les équations deviennent

$$x_{uu} = x_{uv} = x_{vv} = \frac{1}{2\,(u+v)}\,(x_u + x_v).$$

La solution générale de ces équations est

$$x = C_1(u + v)^2 + C_2(u - v) + C_3$$

de sorte qu'en choisissant le repère fixe de façon convenable les équations de notre réseau soient

$$\begin{cases} \varrho\, x_1 = (u + v)^2 \\ \varrho\, x_2 = u - v \\ \varrho\, x_3 = 1. \end{cases}$$

Comme solution de notre problème nous trouvons donc un système de coniques qui ont entre elles une osculation de deuxième ordre en un point fixe P et qui ont comme tangente commune une droite fixe l (faisceau tangentiel osculateur). Par chaque point du plan (à l'exception de P et des points de l) passent deux coniques du système qui sont les deux courbes du réseau sortant de ce point. Du point de vue projectif il n'y a qu'un réseau satisfaisant les demandes de notre problème.

Le réseau trouvé est la projection des asymptotiques de la surface

$$\begin{cases} \sigma\, y_1 = \tfrac{1}{2}(u + v)^2 \log (u + v) - (u - v)^2 \\ \sigma\, y_2 = (u + v)^2 \\ \sigma\, y_3 = u - v \\ \sigma\, y_4 = 1. \end{cases}$$

Cette surface possède donc la propriété de contenir une infinité de réseaux conjugués dont les courbes sont situées dans des plans qui passent tous par un point fixe.

Anatomy. — *On the shape of froth chambers.* By S. T. Bok. (Communicated by Prof. M. W. Woerdeman.)

(Communicated at the meeting of October 26, 1940.)

In the tissues of the human body there is a great variety of froth structures and possibly the spatial networks formed in those tissues by protoplasmic and other rods are related to froth structures in so far that these rods may be found at the place of earlier froth edges. The quantitative investigation of these tissues is ever hampered by the lack of an adequate geometrical analysis of froth structures. We even find the most diverging communications in the literature about the shape of the froth chambers and hence a calculation of the most elementary properties of the shape of a froth — such as for instance the total edgelength in a given volume — is not even possible by approximation. Therefore we have investigated the shape of froth, the general results of which investigation are described in the following pages.

The properties of shape of froth are greatly different, according as the chambers are practically of the same size or not. When the difference in size is great the chambers are all more or less globular: in the spaces left between the larger globes (even when they nearly touch) the smaller globes can find sufficient room without materially affecting the shape of the larger ones. In froths the chambers of which are of the same size, on the other hand, the shape of the chambers approaches that of polyhedrons, i.e. bodies bounded by flat faces. For in that case the chambers are separated by thin, practically flat walls, meeting in approximately straight edges. Only froths with chambers of the same order of size will be discussed here.

To such a froth the two well known laws apply, viz.

1.   three walls always meet in one edge (under approximately equal angles) and

2.   four edges always meet in one point of junction (also under approximately equal angles).

From this it follows that the interfacial angle between two adjacent chamber walls is approximately 120° (i.e. about $^1/_3 \times 360°$) and that the four edges meeting in one point will spatially behave approximately like the four lines connecting the centre of a regular tetrahedron with its four vertices, so that the angle between two intersecting edges will be approximately 109° 28′ 16,......″ (being the angle between the lines in the tetrahedron mentioned, that is an angle with $\cos = -^1/_3$).

The two rules mentioned follow from the conditions for equilibrium in a froth. A

froth consists of two substances which do not mix, either two liquids, or a liquid and a gas, of which the one filling the chambers is disperse, i.e. divided into a number of non-coherent parts, separated by the second, coherent, so continuous, substance, which forms the walls. When the disperse substance fills only a small part of the space we use the term emulsion, otherwise it is a froth. From this it follows that the spatial structure of an emulsion gradually passes into that of a froth, as the relative volume of the dispersed substance (emulsion globules or contents of the froth chambers) increases. The greater the relative volume of the froth chambers, the thinner the walls and the more apparent the froth character. Very thin walls, however, can only exist when the surface tension is great as compared with the specific gravities. The structure of a "fine" froth therefore mainly depends on the surface tension and this is approximately the same in all froth walls. For the surface tension depends on the nature of the two adjacent substances and of the curve radius of the surface. In a froth with a thin wall whose chambers are practically the same size, the greater part by far of the surface is very little curved, only a narrow strip along the edges showing a more marked curvature. So the potential energy, accumulated in the partition planes depends principally upon the nature of the two substances (which nature is the same everywhere), and on the total size of those partition planes. The attempt to establish the equilibrium, which therefore, is mainly an attempt to occupy the smallest possible area, results in a tangential tension in the froth walls, which is approximately the same everywhere. The three walls meeting in one edge exert approximately the same degree of tension, and the three powers (approximately equally great) will only establish an equilibrium if they form approximately equal angles, so that together the three walls must form approximately equal interfacial angles. And where a following froth plane intersects the edge, the interfacial angle between this plane and two of the former being again ca. 120°, the four intersecting lines, the edges, again intersect under approximately equal angles.

In the literature various theories have been put forward concerning certain polyhedrons, which are supposed approximately to have the shape of froth chambers and which, when a great number of them is piled up, are supposed more or less to fulfil the conditions for equilibrium mentioned.

BUFFIN's theory, that this is the rhombic dodecahedron, is the least felicitous, although many after him supported his theory, the last of his followers being SEIFRIZ in 1930. We may derive this rhombic dodecahedron from the situation, found in regularly arranged globes. The regular arrangement (one of the two so-called densest globe cumulations) is characterized by the fact that any 12 globes touching a "centre" globe (all globes having the same size), have their centres in the centres of the edges of a cube whose centre of gravity coincides with the centre of the central globe (see fig. 1, left top corner). If we imagine such regularly arranged globes equally increasing in volume while their centres remain in the same place, they will be flattened against each other in such a way that each plane of contact is found in the place of the tangent plane which, in the original configuration, may be imagined through the point of contact between the two globes. In the case of maximal growth, therefore, they become polyhedrons, which may also be construed in the regular arrangement by construing the mutual tangent plane of two globes through each point of contact. The lines of intersection of those tangent planes are the edges of the polyhedron obtained and it appears that this polyhedron is a rhombic dodecahedron (see fig. 1, right top corner). BUFFIN verified this con-

struction by tying peas in a bag and swelling them in water. The flattened peas frequently had the shape of a rhombic dodecahedron. That this, however, does not correspond to the formation of a froth, is in my opinion
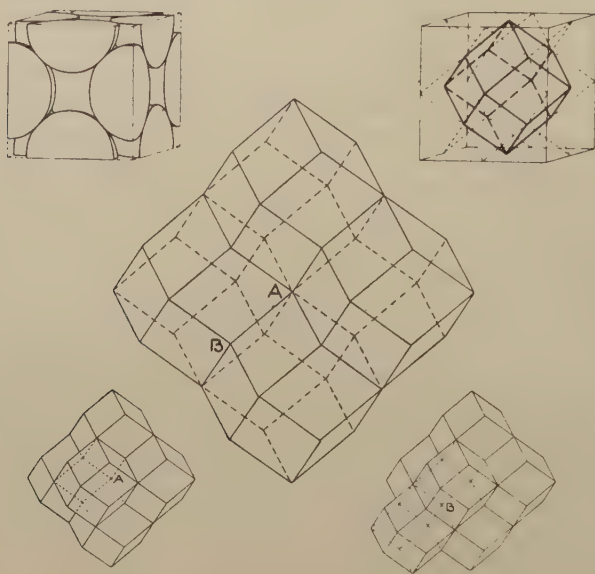
Fig. 1. Regularly piled up globes (see left top corner), when swollen form perfectly fitting rhombic dodecahedrons (see right top corner). In half the points of contact of the edges formed thus 8 edges meet (point *A* in the central figure), in the other points their number is 4 (point *B*). Continued piling up, namely, does not add edges to points *A* and *B*, as may be seen from the bottom figures.

clear from the fact that in this configuration of planes 4 edges meet in only half of all the points of junction. In the remaining points 8 edges meet (two of which always being in line, see A in fig. 1, centre). So this configuration does not fulfil the conditions of the second main rule of a froth (4 edges in every point of junction). And all the froths I examined fulfil this rule. Moreover, in froths I only found very few tetragons as walls and these resembled squares far more than the rhombs of a rhombic dodecahedron, which have angles of 109° 28′ 16,..." and 70° 31′ 43,......". Finally the size of one of these angles varies very much from the ideal value of 109° 28′ 16......". So BUFFIN's thought construction and his experiment do not apply to froths, and this may be understood from the fact that the surface tension did not act as shaping power.

The same error is found in the starting point of Lord KELVIN (1887), who endeavoured to find a body which was to satisfy three requirements, 1. in cumulation the space should be entirely filled, 2. in every vertex 4 edges should meet and 3. the interfacial angles between the planes should be about 120°. The orthic tetrakaidecahedron mentioned by him fulfils these requirements. The two interfacial angles occurring in it are 109° 28′

16,......" and 125° 15′ 25,......". In this train of thought the surface tension does not play a part either and it follows that the result is not applicable to froth. This polyhedron is namely bounded by 8 regular hexagons and 6 squares, and in the froths some hexagons and squares do indeed occur, but they form a small minority. This solution does not fulfil the requirements either that the angles between the edges should be approximately 109° 28′ 16,......": in the orthic tetrakaidecahedron the angles are 90° and 120°.

Both (and similar) reasonings follow a faulty trend of thought: in searching for a polyhedron agreeing as much as possible with the rules of a froth, which on cumulation will completely fill the space. On nearer consideration it is this last requirement which appears to be contradictory to the fact that the shape of a froth depends on surface tensions.

The requirement of complete filling of space namely, implies the idea that the froth walls are flat. In such flat walls the surface tension must everywhere be exactly the same, the curvature being the same (see above) and in order to establish an equilibrium the interfacial angles between the planes must all be identical like the angles between the edges, which ought to be exactly 109° 28′ 16,......". But it is impossible for bodies bounded by flat faces to have only these angles, for the total of the angles of a polygon in a flat plane is always a complete multiple of 180°, and the angle mentioned of 109½° and that of 180° have no common divisor. *So a froth with purely flat walls cannot exist.* Only curving of the faces can solve the problem, on the one hand such a curving being attended with differences in the surface tension on account of which the interfacial angles and the angles between the edges may be somewhat unequal and on the other hand, owing to the curvature, the total of the angles of one face not needing to be exactly a full multiple of 180°. As a matter of fact I found the walls in all the froths examined to be slightly curved, as appeared from the division of light on the glistening froth walls. In figures 3 and 4, two photographs of froth of soapsuds, the edges are seen to be slightly curved. The shape of the froth chambers, therefore differs from a polyhedron with ideal angles, not only in that the angles between its edges as well as its interfacial angles deviate somewhat from the ideal values, but also in that the walls are slightly curved. And that means that the polyhedron does not completely fill the space on cumulation: the interfacial angles of this polyhedron need not be exactly 120° and the three polyhedrons meeting in one edge need not fill the space completely. The only requirements we may make of the approximate polyhedron are those of the two main rules and it is exclusively these requirements which are based on the conditions for equilibrium.

From the properties of a froth we can deduce yet another indication of this polyhedron, although this is not a conditio sine qua non. This indication follows from the consideration that all the chambers adjacent to one specific chamber will in principle have the same relation to that one chamber, there

is at least no a priori reason why one should behave differently in principle from its neighbours. This suggests that in the froth structure there is a certain tendency to make the chamber shape approach to a regular polyhedron.

In my opinion there are four arguments in favour of this being the regular dodecahedron.

In the first place both the interfacial angles and the angles between the edges of the regular dodecahedron have values approaching pretty near to the values mentioned. In the other regular polyhedrons the difference of those angles and interfacial angles is considerably greater, as may be seen from the following table.

	Interfacial angle	Angle between the edges
regular tetrahedron	70° 31′ 44″	60°
„ hexahedron	90°	90°
„ octahedron	109° 28′ 16″	60°
„ dodecahedron	116° 33′ 54″	108°
„ icosahedron	138° 11′ 23″	60°
in froth approximately	120°	109° 28′ 16″

So the interfacial angle of the regular dodecahedron differs not quite $3\frac{1}{2}°$ and the angle of the edges not quite $1\frac{1}{2}°$ from the angles which in flat walls would establish the equilibrium. In all the other regular polyhedrons the differences are considerably greater.

The second argument is that it is only in accumulation of the regular dodecahedron (after modification in the above sense) that 3 planes meet in one edge and 4 edges in one vertex. When regular dodecahedrons are piled up not all the walls can coalesce, a.o. because the three interfacial angles (of over $116\frac{1}{2}°$) which can meet in one edge, leave an opening of ca. $360° - 3 \times 116\frac{1}{2}° = 10\frac{1}{2}°$. But round one regular dodecahedron 12 other nearly regular dodecahedrons can be placed, exactly fitting into the space if some of its interfacial angles are taken a few degrees greater. It appears then that along each edge 3 planes meet and 4 edges in each vertex (compare fig. 2), which corresponds with the figures in the froth structure.

A pile of regular hexahedrons (cubes) will fit perfectly without any modification. But when the vertices coalesce 4 planes will meet in each edge and 6 edges in each vertex, so we do not get the numbers 3 and 4 which invariably are found in froth structure.

When piling up regular tetrahedrons, narrow clefts are always left: the interfacial angle is just over $70\frac{1}{2}°$ and so the five meeting in one edge,

together fill $352\frac{1}{2}°$ which is $7\frac{1}{2}°$ short of the completion of 360°. When we imagine them slightly modified in shape (a.o. by enlarging the interfacial angles to averagely 72°), 5 walls will meet in each edge and 12 edges in each point of junction, which numbers deviate still more from the numbers of the froth structure.
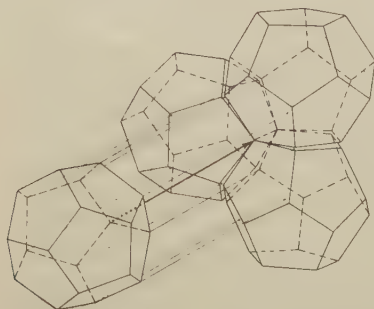
Fig. 2. Four regular dodecahedrons may be piled up nearly fitting round one point.

Regular octahedrons of equal size when piled up also leave clefts (interfacial angle $109\frac{1}{2}°$, so too large for four and too small for five such octahedrons in one edge). Perfectly fitting piling may be obtained by reducing their interfacial angle from just over 109° to 90°. Six such bodies together form a rhombic dodecahedron [1]) and the space may be filled by piling up rhombic dodecahedrons. It is then seen that 4 walls meet along each edge and 8 edges in each vertex, which is again contradictory to the main rules of a froth.

Perfectly fitting accumulation of bodies approaching the shape of a regular icosahedron is not possible. Moreover, in a point of junction of a regular icosahedron 5 edges meet already, which is too great a number to fulfil the condition of the second main rule of froth structure. In accumulation this number would become greater still, and the discrepancy would be even more pronounced.

So the regular dodecahedron is the only regular polyhedron to which the shape of the froth chamber might approach under the condition that 3 planes must meet in one edge and 4 edges in one vertex.

The third argument is given by the consideration that on accumulation of globes of the same size 12 globes can always be in contact with one. On mutual flattening therefore bodies with 12 faces will be formed and if these bodies should approach the regular polyhedron this ought to be the regular polyhedron showing 12 planes, that is the regular dodecahedron.

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[1]) An octahedron has 6 vertices. Through 4 vertices a diagonal plane may be brought, bounded by four edges and having the shape of a square. The 6 octahedrons mentioned may now be placed so that a diagonal face of each coalesces with the wall of a cube. Of the planes within the cube 2 coalesce each time, of the planes outside the cube two are always in line. Together each two planes named secondly form one plane of the rhombic dodecahedron.

Fourth argument: by far the majority of the walls in a froth are penta-gonal and the regular dodecahedron only consists of pentagons, the other regular polyhedrons are bounded by triangles (viz. the tetra-, octa- and icosahedron) or by squares (the hexahedron, the cube).

That most of the walls in a froth are pentagonal is a fact found in each of the froths investigated for this purpose (with chambers of about the same size). Meanwhile a superficial observation of such a froth may easily lead the investigator astray, as the surface chambers have a different shape. When, for instance, we examine a froth obtained by blowing air through a blowpipe into a layer of soapsuds in a cylinder glass (which froth entirely may fill the space of the glass above the liquid) we are at first struck by the lines along which the froth walls attach themselves to the glass wall and nearly all these lines form hexagons of a fairly regular shape. (The same is true of the lines connecting the walls between the upper chambers with the walls separating the chambers from the open air. If looked for, the same picture may be observed on the surface of the liquid at the bottom of the glass). In these surface chambers the conditions for equilibrium are different owing to the fact that on the outer surface of the froth there are only walls running in, not running out. This is the cause that walls between the surface chambers, especially in their most superficial parts, can place themselves pretty truly in such a position that all the interfacial angles are 120°, owing to which a section perpendicular to their edges must show a regular hexagon. And owing to the fact that these walls are perpendicular to the surface, this surface shows such a section.

The influence of the surface being less felt in those chambers that are situated farther away from the surface, we see the general froth structures discussed in this paper, only in the centre of the froth, at least when the cylinder glass is wide enough. In order to make them clearly visible I photographed a froth of soapsuds (made in a cylinder glass) by means of a lens with a comparatively small focus distance (5 cm) and a large opening (F: 3.5), owing to which only the interior gave a sharp picture. It is evident from the photographs that most of the walls are pentagonal (figures 3 and 4).

The preponderance of pentagons I found not only in all the froths consisting of a liquid and a gas, but also in froths of two liquids. Such a froth structure occurs in the yolk of an egg, for instance of a hen's egg. In it the yellow substance lies as a liquid in separate parts of about the same size, separated by walls of an uncouloured liquid. By boiling the yellow liquid coagulates: when carefully pushed or stirred with a small glass stick, a piece of hard boiled yolk, placed in a drop of water on an object glass will separate into a number of coagulated froth chambers. The photographs in fig. 5 demonstrate the fact that these bodies approach the shape of polyhedrons, mainly bounded by pentagons. (I am grateful to Prof. H. G. BUNGENBERG DE JONG for indicating this highly demonstrative example.)
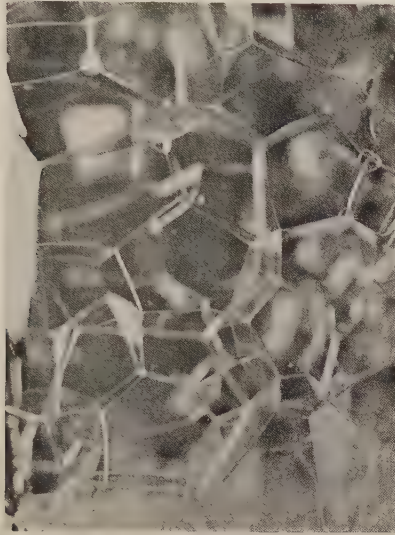
Fig. 3. The majority of walls in a froth of soapsuds are slightly curved pentagons.
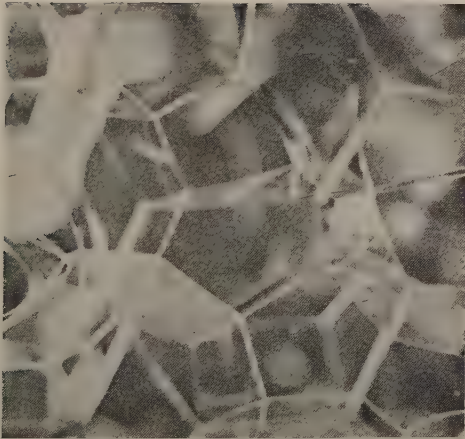
Fig. 4. A few walls in a froth of soapsuds are hexagons, fewer still are tetragons (one of each is seen in the photograph).
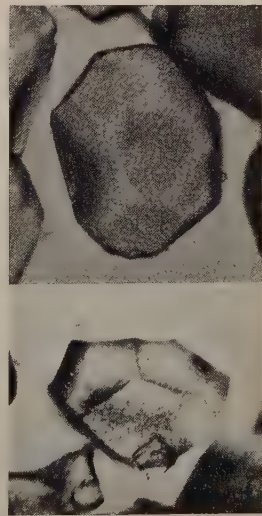
Fig. 5. Coagulated yolk of egg with pentagonal faces.

Meanwhile, when bodies approaching in shape the regular dodecahedron are piled up, a phenomenon occurs, in consequence of which not all the walls can be pentagonal, but in which some hexagons and fewer tetragons must occur, which is also seen to be the case in the froth photographs.

Starting from a regular dodecahedron a rind of 12 other dodecahedrons may be built round the first, which slightly deviate from the regular dodecahedron in the above sense. By each time fixing new edges to each vertex, so that 4 edges meet in each vertex and the edges each time enclose pentagons, we can for an unlimited space of time continue building new rinds round the former ones, all consisting of pentagon dodecahedrons. But the shape of these dodecahedrons will deviate the more from the original, central dodecahedron as they are further removed from it. For it appears that their tangential dimensions (their dimensions in a direction perpendicular to the line connecting their centre with that of the central dodecahedron) ever become greater. The cause of this lies in the discrepancy between the growth of the tangential surface of the rind (proportionate to the square of its radius) and the increase in the number of dodecahedrons per rind. This number is determined by the number and the position of the uncovered pentagons of the previous rind, and it increases more slowly than the surface, so that one dodecahedron must cover a larger part of the rind as the distance from the central dodecahedron increases. If this construction is carried out (I did it with twisted copperwire), the space can ever be taken up by additional pentagon dodecahedrons, but in doing so either the outer dodecahedrons must be larger in all directions, which leaves their shape approximately that of regular dodecahedrons but which increases their volume, or their volume may be left almost unchanged, but then their radial dimensions (along the line running through their centre and that of the central dodecahedron), should be taken increasingly smaller. Then they are more and more flattened, finally becoming broad, thin slices, which cannot exist as froth chambers. In no single way can the space over a large extension be occupied by bodies which are everywhere approximations of regular dodecahedrons of the same size.

This difficulty may be met by placing between the dodecahedrons some bodies with a greater number of planes, in which the interfacial angles need only be changed a little. Without entering into details it may be demonstrated from fig. 6, an arrangement of nearly fitting regular dodecahedrons, enclosing a hexagon. On either side of this hexagon a body is bounded, which on completion of the accumulation would be a tetrakaidecahedron, bounded by 12 pentagons and 2 hexagons. In the same way other configurations produce polyhedrons with a few tetragons. Indeed this occurrence of hexagons and tetragons among the great majority of pentagons is found in the froths. Fig. 4 shows a hexagon by the side of a tetragon. As a rule such a tetragon appears to be approximately a square.

Finally we can discuss briefly the relation between the rhombic dodecahedron, which was supposed to be the result of a regular cumulation of globes and the regular dodecahedron which approaches the actual proportion in the froth.

The twelve globes in a regular arrangement touching some central globe, are not arranged equally round that central globe. If of the cube on whose edge centres lie the centres of the peripheral globes, we select a vertex, we see that 3 globes which touch
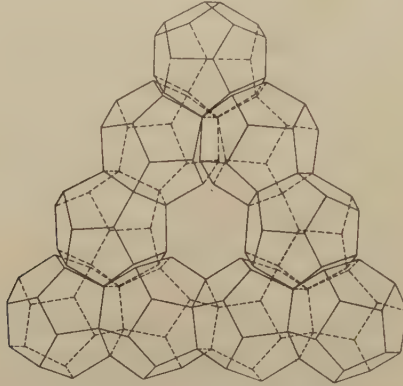
Fig. 6. Regular dodecahedrons can form nearly fitting hexagons.

each other are arranged round it (fig. 7 left). Round the centre of a face of that cube, on the other hand, there are 4 globes, between which a greater space is left open than between the three globes mentioned first. Hence the peripheral globes can be divided in a natural way into 4 groups, each consisting of 3 globes touching each other. Owing to the fact that these three globes also touch the central globe, their three centres and the centre of the central globe lie in the vertices of a regular tetrahedron, being the most stable configuration of 4 globes. When we imagine lines drawn through the centre of the central globe and through each of the centres of these tetrahedrons, these lines also intersect 4 non-adjacent vertices of the cube (see fig. 7 right top corner), owing to which they form mutually equal angles of 109° 28′ 16, …″. They may therefore be conceived as the four trigonal polar axes of the regular system. The three peripheral globes of one group are equidistant from their axis and may be revolved round that axis. They then leave the regular arrangement, to recover it after a rotation of 120°.

In the regular arrangement it is seen that any peripheral globe always touches 4 other peripheral globes. The tangent plane between the peripheral globe selected and the central globe is therefore intersected by the four tangent planes between the central globe and the other peripheral globes mentioned, i.e. the tangent plane becomes a tetragon. The four peripheral globes touching the peripheral globe selected are not arranged regularly round it: they touch two by two, leaving two openings of the same size, situated diametrically opposite each other. The tetragon deducted above is therefore a rhomb. Hence the development of the rhombic dodecahedron. After the revolution round the polar axis described we find other proportions between the peripheral globes. It appears then that any peripheral globe has come to lie between 5 others: so in the case of growth in this configuration the globes will develop into dodecahedrons bounded by pentagons. These are no regular pentagons owing to the peripheral globes being dispersed irregularly round the selected one. But in this configuration the number of contacts between the peripheral globes is considerably smaller than in the regular one (viz. 12 instead of 36), for now only the three globes out of each group touch and there is no contact between globes of different groups. Therefore this configuration, which I propose to call tetratoid (see fig. 7 right bottom corner), leaves room enough for each peripheral globe to relinquish its contact with the other peripheral globes (a peripheral globe is no longer

hemmed in by four other peripheral globes touching it). This opens the possibility that the 12 peripheral globes distribute themselves perfectly equally round the central globe (none of the peripheral globes touching another) and this configuration causes the regular dodecahedron. But the peripheral globes can also distribute themselves *approximately*
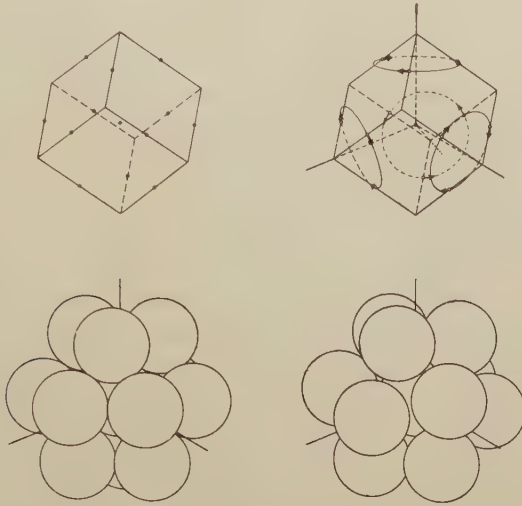

Fig. 7. The regular and the tetratoid arrangement of globes. The figures on the left demonstrate the regular, those on the right the tetratoid arrangement. The top figures show the position of the globe centres with regard to a cube. In the right figure they have revolved from the regular position round the four pictured trigonal polar axes by the length of the arrows (that is 15° in a positive sense).

regularly round the central one, and then the result is the polyhedron which is an *approximation* of the regular dodecahedron with which they have in common that they are both bounded by 12 pentagons.

From this systematic description it follows that the regular arrangement represents only one specific arrangement of the peripheral globes, and one with a maximum of mutual points of contact, which implies a minimum of freedom, and that from this, through rotation on the one hand, and through dispersion of the globes on the other, an infinite number of other positions may develop with fewer points of contact and hence with more freedom. The only other configuration with the maximal number of points of contact is the hexagon (different from the regular one in that out of the groups of three peripheral globes one isolated group has revolved 60° round its axis). Only in these two specific positions the 12 tangent planes are tetragons (in the regular one they are rhombs, in the hexagon they are rhombs and isosceles trapezia), in all the other positions they are pentagons.

The regular and the hexagonal arrangements can be constructed further than the first rind, while the structural plan remains exactly the same. This is not the case with the tetratoid and free arrangements. I have not succeeded in extending them with globes of the same size in such a way that the 12 peripheral globes surrounding any central globe are distributed so that each is surrounded by 5 others. This is in agreement with the fact mentioned above, that the extension with dodecahedrons, bounded by pentagons is not possible without changing the volume or the general shape of those dodecahedrons.

The proposition defended here, that the froth chambers are approximations of regular dodecahedrons has already been put forward by several

investigators. Most of them only adduced the argument that among the regular polyhedrons it is only the regular dodecahedron which has inter-facial angles and angles between the edges approximately agreeing to the angles which develop when 3 flat planes meet in one line under equal interfacial angles and when 4 straight lines meet in one point under equal angles. They did not account for the deviations from those angles and they did not refute the arguments of the authors who mentioned other polyhedrons as approximations of the shape of the froth chambers, so that the uninitiated reader had no arguments for a correct choice. I think I have provided such theoretical arguments in the above, and moreover to have shown that the regular dodecahedron is more in keeping with the actual proportions in a froth than other shapes mentioned in the literature. As for the latter, it was particularly shown that the majority of walls in a froth are *pentagons* (of the regular polyhedron only the dodecahedrons have pentagonal walls) and that among them there occur a few hexagons and tetragons, which must also occur now and then when approximately regular dodecahedrons are piled up.

Anatomy. — *Neuropilem and gliareticulum in the cortex of the cerebrum.*
By S. T. Bok. (Communicated by Prof. M. W. Woerdeman.)

(Communicated at the meeting of October 26, 1940.)

The investigation by R. C. Renes [1]) concerning the abundance of fibres in the cortex of the cerebrum has brought to light that everywhere between the different cells and vessels of the cortex there is the same total length of nerve fibre per spatial unit (except in those places where the myelin sheaths are numerous and so occupy an appreciable part of the space). In other words the neuropilem is equally dense everywhere in the cortex.

This makes it clear how it is that cortex areas, in which there occur mainly small nerve cells, are much richer in nerve cells than areas with larger nerve cells (by which we mean that in areas with small cells the number of nerve cells per spatial unit — the density of the nerve cells — is greater than in areas with large cells). For generally a larger nerve cell has a larger dendrite complex with a greater total length of dendrites than a smaller cell and because the richness in fibres is the same everywhere, this greater length of dendrites requires more space, so that the space between the nerve cells must be greater as those cells are larger.

Meanwhile Renes' investigation has not yet made it clear through what cause this distribution of fibres is the same everywhere in the cortex.

The constancy found in the length of the nerve fibres per spatial unit suggests that everywhere the fibres have reached their maximal development, that, as it were, the cortex is everywhere filled to the greatest possible extent with nerve fibres. But what is the factor, determining the maximum which brings to a close any further development of ever new branches between those existing already?

This is certainly not the mere fact that the space between the nerve fibres should have become too narrow to allow fibres of sufficient thickness to penetrate: in the Bielschowsky-preparation everywhere between the thin black-impregnated fibres there is a space which is averagely considerably more than ten times as great as the average thickness of those nerve fibres, so that certainly hundreds of times more fibres of the same size might find a place in the same space in the cortex.

Renes had investigated if, perhaps, the cause may be that there must always be a certain minimal distance between the nerve fibres. If such a minimal distance is indeed prescribed, the cortex space would be "filled up"

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[1]) R. C. Renes, Over de vezelrijkdom in de hersenschors. Thesis Leyden 1940.

much sooner (then the maximal length of nerve fibre per spatial unit would be much smaller) than if the fibres are allowed to touch. For this purpose he has measured the distance between neighbouring nerve fibres (= the length of their common perpendicular) and in one respect his result seemed to confirm this idea.

He has namely made these measurements in different architectonic fields and in each field separately in layers II—VI (layer I is not sufficiently impregnated in the BIELSCHOWSKY-method). In each field a small area was selected in each of the layers and in each of those areas the fibre distance was measured in a hundred cases. It was seen that in each area examined these distances vary greatly in size, but in all areas their size varies between practically the same minimum (0.4 μ) and about the same maximum (1.9 μ). So in each field he found a certain minimal fibre distance and this minimal distance appeared to be the same in all the areas examined (for the mutual variations between these minima practically remained below the value of the error of measurement: the average error of the individual measurements was determined at 0.027 μ and in the various areas examined the smallest of the distances found varied between 0.40 and 0.44 μ, once it was 0.46 μ). So it appeared that there is indeed a minimal fibre distance throughout the cortex, which was found everywhere, and within which no two nerve fibres cross each other.

But in two important points the results of his measurements are by no means in agreement with the theory tested.

In the first place the distribution of the various fibre distances was not in agreement with the idea that everywhere in the cortex there is the maximal quantity of nerve fibre which there might be if it was only limited by a minimal distance between the fibres. If this maximum should be reached it might be expected that in many places the fibres would cross at the minimal distance. It is true that greater distances would also occur owing to two fibres crossing at a certain place while they had already been fixed in place through neighbouring crossings, with minimal distance, but the greater distances would become less frequent as they became greater. A frequency curve of the different fibre distances found, would therefore coalesce with the zero line in values less than the minimal distance, at the minimal distance it would rise vertically to the top value after which it would slowly fall to the greater values.

The frequency curve made of the fibre distances found does not fulfil this expectation. After the minimal value of 0.4 μ it does, indeed, climb steeply, to fall slowly after reaching the maximum, but the top lies too far past the minimal value (namely at 0.56 μ) for this deviation to be attributable to errors in measuring. Only with an average error of measurement of 0.11 μ the vertical truncation of the curve in these circumstances would have been flattened to approximately the shape found in the actual measurements. But repeated measuring of the same distance of intersection proved the error to have been no more than 0.027 μ, so that

we are justified in attributing a real significance to the gradual climbing of the frequency curve between 0.4 and 0.56 μ. So the distribution of the measurements found is not in keeping with the idea of a maximal filling limited by a minimal distance between the fibres prescribed.

In the second place the value of the minimal distance is contradictory to this idea: if the fibres could not come closer together than the minimal distance of 0.4 μ, the total fibre length per spatial unit could be many times greater than the length found. Neglecting the thickness of the fibre and supposing all the fibres to run parallel (in order to obtain maximal "filling") then a simple calculation shows that per 1000 μ^3 of space there would be 14450 μ of fibre, that is 72 times the length found (200 μ). Owing to the thickness of the fibres this value will probably be a little less, but not much, because the average fibre thickness may be estimated at less than $\frac{1}{4}$ the minimal distance between the fibres, and the total fibre length per spatial unit would be less than 4 % smaller. A greater influence has the fact that the fibres do not run parallel but in many different directions. Owing to this factor the possible length will be a few times smaller than calculated above, but it will certainly be dozens of times greater than the one found in the cortex.

So RENES' calculations show that the "fibre filling" is the same everywhere in the cortex, but they do not throw light on the factors which keep the filling constant. In particular they deny the supposition that there would be here a maximum of fibres, limited only because they must not or cannot approach any nearer than to a certain constant distance. So the constancy of the fibre filling must be owing to some other cause. What cause can that be?

The reconstruction of a small fibre area (measuring 15 × 15 × 15 μ), made by RENES — a stereoscopic picture of which is given in his thesis — showed that the fibres are not distributed equally about the space, but that comparatively large open spaces are found, between which the fibres are packed as in thin membranes or in rods. These open spaces are of about the same size, the centres of two membranes facing each other are about $5\frac{1}{2}$ μ apart. So it is probable that the occurrence of these open spaces limits the quantity of nerve fibres and the question arises if the constancy in the fibre filling may be a consequence of a constancy of these open spaces.

About the nature of these open spaces we can state with certainty that they do not consist of cell bodies: they do not contain nuclei. Moreover they are far too numerous for that: from the measurement given it would follow that there ought to be about 9000 in an area of 100 × 100 × 100 μ, whereas there are only 80 to 120 cells (glia and nerve cells together).

The order in size of these open spaces corresponds to the meshes of the glia reticulum as drawn by SPIELMEYER, by HOLZER, and others. The supposition is obvious that they are vacuoli, between which protoplasm walls or protoplasm rods form the bedding for the nerve fibres.

Thus far it has not proved technically possible to obtain an insight

through microscopic observation into the shapes of these vacuoli and protoplasm walls or rods. A purely theoretical deduction, however, reveals some remarkable circumstances, which suggest a very simple way in which the fibre quantity in the cortex would be limited to the measure found by these very vacuoli.

We may start from the provisional supposition that these vacuoli are non-coherent; then the protoplasm would lie between them in the shape of a froth, and then the above measurement of the froth chambers tells us, according to a calculation which will be given below, that the total edge length in this froth corresponds with the length of nerve fibres found (200 μ per 1000 μ^3). If the vacuoli should be more or less coherent this agreement would still hold, if only the protoplasm walls were penetrated, as in a sponge, the protoplasm rods being left in the place of the original edges of the froth structure. The fibre filling found would then be determined quantitatively by the simple rule: in one protoplasm rod one nerve fibre.

In the calculation indicated above the open spaces were conceived as chambers in a protoplasm froth and the reconstruction showed that these chambers are about equally large. In a communication about the shape of froth chambers [1]) I could determine that in a froth with about equally large chambers the latter approach the shape of a regular dodecahedron. For the calculation we therefore started from the conception that when we imagine flat planes between the open spaces these planes would be the walls of regular dodecahedrons. This conception only approaches the actual proportions, as has been explained in the communication mentioned, but for our purpose this approximation proves amply sufficient.

In the reconstruction the centres of neighbouring open spaces lie about 15 cm apart. As the enlargement is 27500, they are about 150000 : 27500 $=$ $= 5.46$ μ apart in the preparation, so that the radius of the inscribed globe of the dodecahedrons may be calculated as $\frac{1}{2} \times 5.46 = 2.73$ μ, from which follows the edge length as 2.46 μ [2]) and the contents 114 μ^3 [3]).

Each regular dodecahedron is enclosed by twelve pentagons, meeting in $12 \times 5 : 2 = 30$ edges. In the cumulation 3 dodecahedrons meet in one edge, so that per dodecahedron there are $30 : 3 = 10$ edge lengths or $10 \times 2.46 = 24.6$ μ of edge length. Per 1000 μ^3 there are 1000 : 114 $= 8.75$ dodecahedrons, giving a total of $8.75 \times 24.6 = 215$ μ edge length. In the same volume RENES found 200 μ of fibre length.

Considering that the conception of a froth as an accumulation of regular dodecahedrons is only an approximation, this agreement is so great as to warrant the conclusion that practically there is as much fibre length in the

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[1]) See these Proceedings, p. 1180—1190.

[2]) From the radius $r = 1/20\,a \sqrt{250 + 110 \sqrt 5} = 1.11\,a$ or
$$a = 0.9\,r = 0.9 \times 2.73 = 2.46\,\mu$$

[3]) From the contents $c = 1/4\,a^3\,(15 + 7\sqrt 5) = 7.663\,a^3$
$$= 7.663 \times 2.46^3 = 114\,\mu^3.$$

neuropilem as the edgelength would be, if the cortex was a froth with a chamber size agreeing with the data of the reconstruction.

As the distance between the nerve fibres has been measured, it is attractive to use the froth structure with averagely 8.75 chambers per 1000 μ^3 as a provisional model, and to ask ourselves what would be the aspect in that model of the crossing of the fibres, of which there is each time one in an edge.

In a froth four edges always meet in one point of junction. A nerve fibre which we imagine in one of those edges will lie in another of those four edges after passing that point: so it occupies two out of the four edges of that one point of junction. Hence in the two remaining edges there will · be a second fibre, which passes the first in this point of junction. According to this model there can only pass two fibres through the same junction, otherwise there would be more than one nerve fibre in one rod.

It is clear that the two fibres would intersect each other in this point of junction, if the edges as well as the fibres were mathematical lines without thickness. But in our model both have thickness. The mathematic-ally conceived planes between the adjacent chambers discussed above served only to calculate the edgelengths, in our model these planes lie in a protoplasm wall of a certain thickness and thus the abstractly conceived mathematical edges also lie in protoplasm rods with a certain thickness, while the junctions where four rods meet are also protoplasm areas of a certain spatial extension. If this extension is large enough as regards the thickness of the fibre the two fibres can cross here and leave a certain distance between them. Then they lie no longer accurately at the site of the mathematical edges, but a little by the side of them, at least in these junctions where they cross one another.

The four edges meeting in one point of junction in a froth, together form averagely equal angles, that is to say, they run from that point like the lines connecting the centre of a regular tetrahedron with its four vertices. If a nerve fibre was situated exactly in the mathematical edge that fibre would make an angle at each point of junction it passes (viz. one of averagely just over 109°). If we set out to draw in our model the shortest fibre between two far removed points, which fibre was everywhere to coincide exactly with the mathematical edge, then this would not be a straight line but a broken one, making the angle mentioned at regular intervals, but otherwise approaching the straight line mentioned. It is then seen that this broken line is wound screw-wise round a nearly straight axis (apart from the fact that it consists of a number of straight parts, not being curved, therefore, like a mathematical screw line).

Because the nerve fibres in our model — at least in the points of junction of the edges — should not coincide exactly with the mathematical edges, the possibility arises to truncate and round off the angle which each fibre would otherwise have to make at each point of junction. A fibre thus rounding off every angle, will by this fact be a little shorter than the total

of the mathematical edges it follows and the screw-like line mentioned will approach to a straight line. Maximal shortening would be obtained by such a fibre which would entirely flatten all its angles: such a fibre would be situated in the axis of the screw-wisely broken line of the corresponding mathematical edges. The question may be asked, what would be the distance between two such maximally straightened out fibres.

A purely mathematical treatment of this problem is not possible, as the froth chambers have not the exact shape of a regular dodecahedron. The property discussed, that regular dodecahedrons cannot be piled up exactly fitting, is the cause that the froth chambers must differ a little from regular dodecahedrons, and in the article mentioned on the shape of froth chambers it has been explained that on account of the surface tensions in these shapes, these deviations must a.o. consist in slight curvature of the walls and of the edges, which curvings differ from edge to edge and from chamber to chamber. Meanwhile it is possible in models of regular dodecahedrons to indicate, approximately at least, where these lines must be situated, and it is then seen that the distance of intersection between these axis lines in each point of junction is approximately 0.75 of the edge length. In our model, where the edge length represents 2.46 μ, this gives a distance of intersection of 1.85 μ and this is the maximal distance which these fibres can have with complete straightening out of their spiral course. This theoretically derived maximal distance of intersection of 1.85 μ comes fairly near to the maximal fibre distance of 1.9 μ, prevailing in the measurements, so that our model appears readily to account for the maximal fibre distance found by RENES.

Yet another consequence to be derived from the model agrees with a peculiarity of the fibres in the preparations, qualitatively as well as quantitatively.

If the maximal fibre distance was really determined by the above straightening out of a screw-like course to a straight one, then only those few fibres in which the maximal mutual distance of 1.9 μ had been found would entirely have given up their screw-like course, while all the other fibres had not quite straightened out to a straight line. So the many fibres should have a screw-like course with a pitch prescribed by our model. When we follow those mathematical rods which wind round a straight line, it is seen that the axis line is again on the same side of those rods if along that axis line a distance has been covered which is a little smaller than the distance between two adjacent dodecahedron centres. According to the model the pitch should therefore be a little less than 2r or 2 × 2.73 μ, so a little less than 5.46 μ.

As a matter of fact most fibres in a BIELSCHOWSKY preparation do not run straight but with fairly regular undulations. In a fibre which lies about in the direction of the optic axis of the microscope, and which therefore, when the micrometer-screw stands still is seen as a black point (small spot), we see in most cases that this point moves in an approximately circular

direction when the micrometer-screw is turned: the undulation proves to be a consequence of the screw-like course. The pitch of this screw can be easily measured in those fibres which run parallel to the section plane. In the drawings RENES made to determine the length of the fibres, their undulation is plainly visible (see fig. 5 of his thesis) and from it the average pitch of 5 μ was measured. This measurement also lies in the same order of size as the measure of a little less than 5.46 μ theoretically derived above.

In order to get the nerve fibres in the junctions of the protoplasm rods to cross at the mutual distances found, these junctions must have special dimensions and the question arises what shape the protoplasm froth should have in connection with this requirement. When calculating this we are up against the uncertainty whether the nerve fibres can lie quite peripherally in their protoplasm layer, or if they have always to be surrounded on all sides by a layer of protoplasm of a prescribed thickness. On the ground of the minimal distance of intersection of 0.4 μ, RENES already pronounced the supposition that every nerve fibre must be surrounded by a protoplasm sheath 0.2 μ thick. Starting from this supposition, a calculation, which will be published in detail elsewhere, shows that in order to obtain the maximal distance towards the neighbouring one of 1.9 μ the junction must rather exactly have the shape that would arise if the adjacent froth chambers were globular and just touching. Where, on the other hand, the chambers are more or less flattened by contact, the junctions would be such that the distances of intersection would have to be smaller than the maximal value of 1.9 μ. The fibres crossing in these smaller junctions could not have a perfectly straightened out course. The variation of the distances between the fibres might therefore — in part at least — be based on a variation in the thickness of the froth walls (the protoplasm rods respectively). In this connection it is remarkable that already with very slight flattening of the froth walls (a very small penetration respectively) the protoplasm rods between the junctions would become too thin to contain more than one nerve fibre with its sheath 0.2 μ thick.

So here the froth structure also acts as a factor determining that per protoplasm rod there is as a rule only one nerve fibre.

The supposition that the protoplasm of the cortex is constructed like a froth (or spongelike as a network of rods, taking the place of the edges in that froth), the chamber size of which corresponds to the dimension of the open spaces in RENES' fibre construction, produces consequences which qualitatively and quantitatively correspond to the following properties, which had come to light through the measurements taken of the nerve fibres:

1. The total length of protoplasm rods (froth edges) is practically equal to the total length of nerve fibre, viz. 215 μ and 200 μ respectively per cell-free cortex area of 1000 μ^3;

2. The distance between two perfectly straightened out fibres, crossing

each other in the junction of the protoplasm rods (1.85 μ) practically agrees with the maximal value of the distance between neighbouring nerve fibres in the cortex (1.9 μ);

3. According to the supposed froth or reticulum structure of the protoplasm, the fibres which are not entirely straightened out (the great majority) ought to describe screw-like lines with a pitch of not quite 5.46 μ, while most of the nerve fibres in the BIELSCHOWSKY preparation do indeed follow screw lines, with a pitch of averagely 5 μ;

4. According to a calculation not included here for lack of space, the complete straightening out of the fibres with its corresponding fibre distance of 1.9 μ could occur only in such places where the chambers are not flattened by contact with each other. If the chambers are a little larger (everything calculated with a constant mutual distance between their centres) the flattening (or penetration) of the frothwalls must be attended with screw-like fibres and smaller distances between the fibres. The protoplasm rods between their junctions will then also become so thin that they can contain only one nerve fibre with its layer of protoplasm 0.2 μ thick. As we saw above, this is indeed the rule.

It stands to reason that the protoplasm froth or reticulum may be identified with the so-called glia reticulum, the reticulum of protoplasm rods, into which the extreme ends of the differentiated gliacells finally pass and which — according to the drawings occurring in the literature — agrees in order of size with the open spaces in the reconstruction of RENES.

According to the theory discussed in the above, the relation between the unmyelinated nerve fibres and the protoplasm in the cerebral cortex is the same as in the peripheral nerve system.

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Anatomy. — *Some topographic relations of the orbits in Man and Anthropoids during ontogenesis, especially bearing on the ontogenetic development of the "rostrum orbitale"*. By J. Ariëns Kappers. (Communicated by Prof. H. M. de Burlet.)

(Communicated at the meeting of October 26, 1940.)

When examining the base of an adult human or anthropoidal skull we see how the two roofs of the orbits protrude more or less into that part of the skull cavity in which the frontal lobes of the brain are situated. Between these roofs a more or less deep space exists at the bottom of which the lamina cribrosa is found. Therefore the anterior part of the base is not flat in a transverse sense either in Anthropoids or in Man. That part of the frontal brain lobes, including the lobus olfactorius, which lies between the orbital roofs is commonly called rostrum orbitale. It will be our aim to discuss in Man and Anthropoids the form and as far as possible the genesis of the space between the orbital roofs in which the rostrum lies.

Looking at the well-known model of the primordial cranium of a human embryo of 8 cm crown-rump length, made by Ziegler after Hertwig, we see how the ala orbitalis forms a nearly flat plate extending at right angles from the septum of the nose, so that the primitive roofs of the orbits practically do not rise above the level of the lamina cribrosa and the base of the skull is nearly flat in a transverse sense. The same is seen in the frontal section through the head of a human embryo of about two months old reproduced in figure 1. The section is not quite perpendicular to the aequatorial plane, so that the right side of the embryo, to the left in the figure, with which we will deal only, is cut more backward than the left side. The septum nasi protrudes above the level of the roof of the nose cavity and, what is not the same thing here, above the bottom of the fossa cranii, forming the crista galli at the anterior end of the skull base. On the left side of this crista in the figure the base of the skull, i.e. the roof of the orbit, slopes slightly laterally and upwards to the ascending wall of the brain cavity, being still nearly flat. Nothing is yet to be found of an elevation of the roof of the orbit protruding into the brain cavity.

No indication is seen of an *U*- or *V*-shaped space between the roofs of the orbits, in which in the adult stage the rostrum orbitale lies.

We now compare this section with figure 2, showing a frontal section through a senile human skull sawn perpendicularly to the lamina cribrosa,

through the forepart of the crista galli. In this way the two sections to be compared are as homotopic as possible.
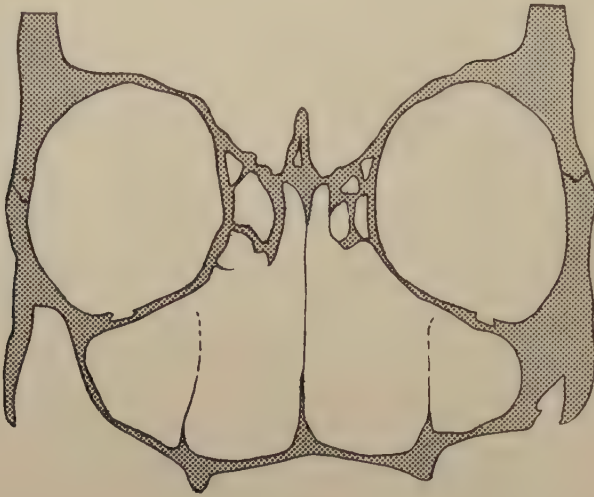
Fig. 2. Frontal section through senile human skull.

We chose this adult skull because in this specimen the pit beside the crista galli in which the olfactory bulbs are situated and the whole U-shaped space between the roofs of the orbits were so pronounced that one could speak of a real fossa interorbitalis encephalicus in which the rostrum orbitale lies.

Apart from some very similar relations in the two sections, we observe also a remarkable difference.

First we will point out the most important, quite similar relation between the position of the orbit as a whole towards the cavum nasi in the two sections. Connecting in thought the centres of the orbits, as done by BOLK, we see that this line in the two skulls not only runs beneath the upper border of the crista but even a little underneath the roof of the cavum nasi. BOLK, using a frontal section through the skull of a neonatus, laid great stress on this point, saying that also in this respect the adult human skull maintains its fetal features. He used for comparison a new-born and an adult skull. That we find the same state of affairs in a so much younger skull corroborates his point of view.

Passing to the differences which exist between the two sections, we may first call attention to figure 3, showing a frontal section through the skull of a human fetus of about the 8th month perpendicularly to the lamina cribrosa. A remarkable difference with regard to the young embryonic as well as to the adult skull, appears in the vertical height of the orbit relatively to that of the cavum nasi. In comparison with the two months old embryo, the vertical height of the orbit is increased very much relatively to that of the nose cavity. The line, however, connecting the centres of the two orbits would have the same position with regard to the tectum nasi.

This implies that the roof of the orbits has grown upwards, while the bottom has grown downwards. In this way the roofs are beginning to protrude into the cavum cranii, forming between them the U-shaped valley
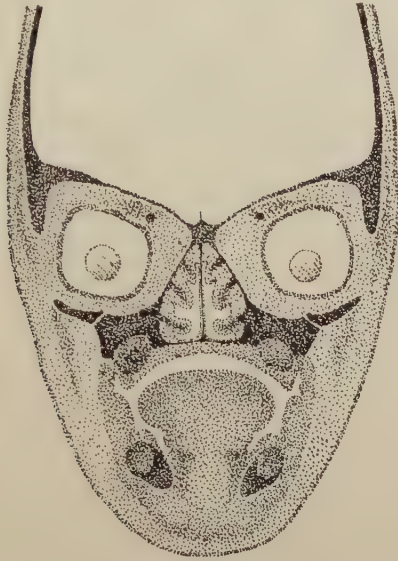
Fig. 3. Frontal section through head of human fetus aged 8 months.

in which the rostrum orbitale lies instead of the flat plate existing in the younger embryo. Later on in the ontogeny the nose cavity grows downwards, as is well known, and beside it and underneath the orbits are then formed the maxillary sinuses. That is why in the adult skull the relation between the height of the orbit and that of the cavum nasi on this level nearly reaches again the early embryonic value. Thus the protrusion of the orbital roofs in adult Man which does not exist in early embryonic development, is primarily due to a vertical enlargement of the orbit relatively to the nose in a fetal stage of development. However, the relative position of the centres of the orbits to the tectum nasi remains the same during development and therefore the position of the orbits in adult Man, being quite lateral to the nose cavity, is an ontogenetically conservative one.

Passing now to the Anthropoids it is well known that in the adult stage the U- or V-shaped "valley" between the orbits gives the impression of being much deeper here than in Man.

To get some information about the position of the orbits relatively to the nose in very young, fetal stages of development in Anthropoids is very difficult owing to the lack of material. Descriptions of anthropoid embryos are very rare and as far as we know sections through such embryos were never published. In 1885 DENIKER dissected a fetus of Gorilla and one of Gibbon and gave a very elaborate and thorough description of the whole

and the parts. As far as the whole of its features were concerned, the Gorilla fetus was at the same stage of development as a human fetus of the 5th—6th month. The Gibbon fetus was older and probably in its last intra-uterine month. About the frontal bone of the young Gorilla DENIKER says:

"La lamelle de l'os frontal, formant par sa face inférieure la voûte de l'orbite, est excessivement mince; elle n'est point encore bombée vers la cavité cranienne comme chez les gorilles même très jeunes. Par suite de cette disposition, la lame criblée de l'ethmoïde, encore cartilagineuse, se voit très nettement, tandis que, dans les crânes même les plus jeunes, elle est enfoncée profondément entre les frontaux et à peine visible."

This last statement, where he deals with the relation in Gorilla infants, may be slightly exaggerated, but what he says about the conditions in his Gorilla fetus is clearly visible in his figure of the skull base. The frontal part of the base, upon which the frontal part of the brain is situated, is very flat indeed and has not at all the shape which it has in postnatal stages. Here also the orbits are situated laterally to the nose cavity. Therefore nothing of a rostrum is seen on his figure taken from the lateral side of his endocranial cast of the Gorilla fetus.

Speaking of the skull of his Gibbon fetus, he says:

"La lame criblée cartilagineuse a la forme ovoïde... les deux os frontaux se touchent presque en arrière et en avant de la lame criblée; chez le Gibbon adulte, ils vont se souder en cet endroit, et c'est à peine si l'on apercevra la lame criblée, enfoncée entre les deux os qui deviendront très bombés".

Looking at DENIKER's figure of the skull base of the Gibbon fetus, we see that the anterior part is somewhat more "bombé" than in the fetus of Gorilla and not quite so flat. Yet, there is still a very great difference in this regard in comparison with the adult stage. That it is still more flat in the Gorilla is probably due to the lesser age of the latter.

In figure 4, we show an X-ray photo of the skull of a Gorilla fetus, present in the collection of the museum of the Anatomy Department of the University of Amsterdam, the exterior of which was formerly described by BOLK. This specimen must have lived till the end of pregnancy. The orientation in the photogram is not far from the Frankfurt plane being perpendicular to the plate.

Here the orbital roofs are already vaulted, so that the formation of an orbital keel has begun, but it is curious to see how the line connecting the centres of the two orbital adituses would not run far above the brain base and the roof of the nose, quite contrary to the state of affairs in most adult Anthropoids, especially in Gorilla (see figure 6). This also resembles quite human conditions.

Descriptions of the primordial crania of Primates other than Anthropoids are rather rare. In 1902 FISCHER published and described a model of the primordial cranium of a Cercopithecus cynomolgus embryo of 25 mm crown-trunk length. His figure clearly shows that boths alae orbitales are remarkably flat, so that here also the orbits were lying towards the sides of the lateral nose walls and a rostrum orbitale must have been still absent.

FISCHER himself says about this region: "Die vordere Partie der Ala orbitalis (also damit auch die Cartilago spheno-ethmoïdalis) ist völlig eben, liegt auch etwa in einer Flucht mit der Lamina cribrosa, genau wie beim menschlichen Embryo, während bei den meisten erwachsenen Affen das Dach der Augenhöhle (Frontale) sich so stark gegen das Gehirn vorwölbt, dass die Siebplatte in eine tiefe Spalte zu liegen kommt." In a later article this author describes the model of the same specimen once more together with that of a Semnopithecus maurus embryo of 53 mm crown-trunk length. Speaking about the ala orbitalis he says here: "Sie ist als eine mächtige Platte entwickelt, die sich von der Lamina cribrosa und den Sphenoïdbalken aus als Dach der Augenhöhle seitlich ausspannt. In ihrer Form und Lage stimmen Affe und Mensch völlig überein, weichen aber von andern Säugern ab." The solum supraseptale of Reptiles, after GAUPP the homologon of the ala orbitalis, is a flat plate which rises in an oblique position lateraly to the median plane, whereas in Primates the position became nearly horizontal. Further on FISCHER says that in the Macacus cynomolgus embryo the upper sides of the alae lie in one plane with the lamina cribrosa, whereas in Semnopithecus the latter is already somewhat sunken between the orbital roofs. This also appears from the figures with which he illustrates his article. The difference between the two monkeys on this point, in our opinion, may be caused by the fact that the embryo of Semnopithecus was somewhat older, thus approaching the adult relations.

From these facts it appears that in the younger fetal stages of many, if not all, Primates the topographic position of the orbits relative to the nose cavity is the same: the plane through the orbital axes lies beneath the subcerebral part of the nose roof, in other words: the orbits are situated beside the nose cavity, and there is no rostrum orbitale, the skull base being nearly flat.

To get a more objective idea of the topographic position of the orbital roofs in our own postnatal material we studied quasi-frontal sections through the skull base.

To avoid a further destruction of our skulls of a great many Anthropoids and Men of different ages which for other purposes had been sawn along the median plane, we made frontal figures with the aid of MARTIN's diagraph of the inner wall of the skull cap and the skull base on the level of the posterior border of the lamina cribrosa of one half of the skull, the skull being oriented with the Frankfurt plane perpendicular to the drawing paper. After we had in this way got the relations on one side of the skull, the whole outline of the inside of the skull on this level could be easily reconstructed, supposing that the skull was symmetrically built. A so far occipital level was chosen, and not for instance the level of the frontal border of the lamina, because of the soon receding frontal bone in the Anthropoids. Starting our drawing in the mid-line on this level and in this orientation, we soon should reach the fronto-lateral side of the skull cap with the point of the diagraph without touching the orbital roofs at all.

The drawback of our method is that the point of greatest height of the orbital roof above the roof of the nose is not always fixed in the figure. This disadvantage was removed by projecting that point in the level of our drawing by means of the diagraph. It was not necessary to note also the median point of the nose roof on this more frontal level, because in this orientation the part of the roof between the two levels was almost perpendicular to the drawing-paper.

In figure 5 we give the drawing of an infantile and an adult specimen of all genera of Anthropoids, excepting Siamang. Of adult Man only we give two figures to show the possibility of individual variations in this

region. It appears that the outline of the inside of the cavum cranii at this level in Man has a much greater extension than in the Anthropoids, the part of the frontal lobes above the orbital roofs being absolutely much
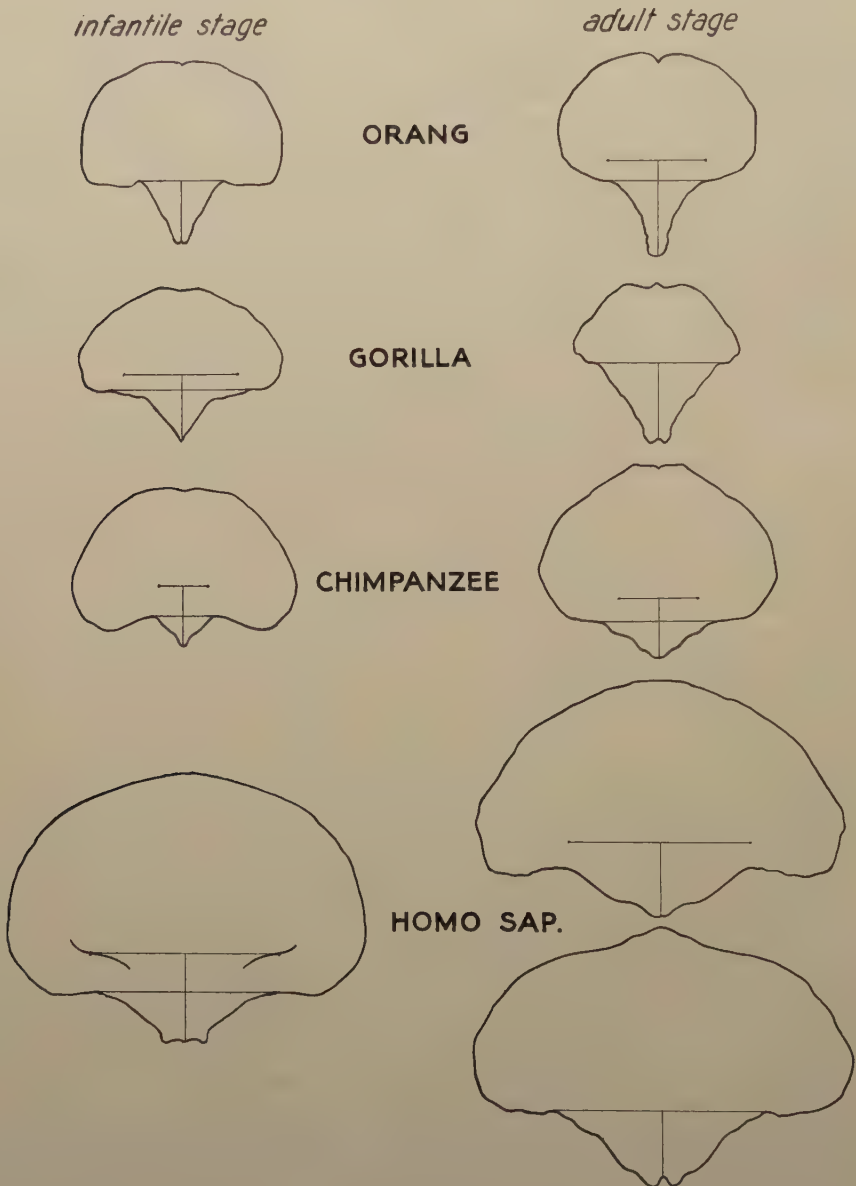

Fig. 5. Comparison of the development of the rostrum orbitale and the relative topography of the orbital roofs in young and adult Summoprimates. (For explanation see text). In the infantile Orang teeth shed was just beginning. The young Gorilla had a milk dentition which was not yet quite complete while in the Chimpanzee child a nearly complete milk dentition was present. The human infant was aged 1.5 years. All drawings ½ natural size.

broader and also higher than in the Anthropoids. The latter is partly due to the fact that in the large apes the frontal bone recedes very soon above the orbits or, in other words, the forehead is really receding, though not so much as is suggested by the big torus supra-orbitalis. A further relative difference at this level between the cavum cranii of Anthropoids and Man is the greater breadth relative to the height in the latter.

Passing to the fossa interorbitalis, containing the rostrum, we cannot demonstrate a clear postnatal ontogenetic difference in each genus separately, but we should remember that the infantile skulls used were not very young. An apparent fossa already occurs in each infantile skull. The impression of the depth of the fossa is determined by two factors, viz. the breadth and the real depth, i.e. the length of the perpendicular between the line connecting the highest points between the two roofs and the basis cranii in the mid-line, as demonstrated in the figures. If the distance between the highest points of the orbital roofs becomes greater, while the real depth of the fossa remains the same, we get the impression that the fossa is less deep than before.

It now appears that the absolute depth of the fossa measured between the roof of the nose and the line between the two highest points of the orbital roofs is not so much greater and at times even less in the Anthropoid skull than in that of Man. It is only the breadth of the fossa, measured between the two highest points, which is always much greater in Man and this is the principal cause why in the latter the fossa seems to be less deep. In fact it is only much broader than in the Anthropoids. The narrowness of the fossa in Apes makes it much more pointed dwonwards, thus justifying the name of rostrum orbitale or "bec encéphalique" for the part of the frontal brain situated in it. In Anthropoids half of the breadth and often even the whole breadth, is always less than the height, in Man on the contrary this breadth always exceeds the height.

Summarising we can say that the impression of the relatively greater depth of the cavity between the orbits in Anthropoids, in which the rostrum orbitale is situated — taken by neurologists when found in Man for a primitive feature —, in the main is rather due to the smaller distance between the orbits, i.e. the narrowness of the rostrum, than to an absolutely greater depth in comparison to the conditions as found in Man. Nevertheless it is true that the depth of the U- or V-shaped valley between the orbits in Anthropoids is greater relatively to the total base-vertex height on this level and this fact adds strongly to the impression of a greater rostral development in Anthropoids.

Discussion.

When tracing the ontogenesis of the U- or V-shaped valley between the orbital roofs in which the rostrum orbitale is situated, we saw that it is already present in young infantile stages of all Summoprimates and that

its morphological alterations during postnatal development are only very small and not essential. Yet, this fact does not imply that the orbital roofs protrude during the whole of ontogenesis into the anterior part of the brain cavity. On the contrary, it appeared that in prenatal development of Primates there is a stage in which the roofs of the orbits in the primordial cranium, and still later on, lie nearly in one plane with the lamina cribrosa, the subcerebral roof of the nose. At this time nothing of an orbital keel is to be found, either in Apes or Man. Somewhat later, in Homo perhaps in the third month of pregnancy, the roofs of the orbits begin their vaulting and herewith their protruding into the anterior part of the brain cavity. This is a fact which hitherto has not been sufficiently realised and which has a certain value for the understanding of the phylogenetic laws and differences in this region between Man and Anthropoids.

Till now it was commonly thought that the rostrum in Man is absent or far less developed than in Anthropoids, owing to the greater development of the frontal part of the brain in Man. Thus BOULE and ANTHONY, speaking of this "bec encéphalique" say:

"Chez l'homme normal au contraire (to the Apes), par le fait du développement des lobes frontaux, aussi bien dans le sens latéral que dans le sens vertical, la région cérébrale antérieure n'offre pas cet amincissement, et les parois supérieures des cavités orbitaires tendent à s'aplatir et à se placer sensiblement dans un même plan, le bec encéphalique est très reduit."

Consequently they try to explain the lack of a rostrum in Man in a simple mechanical way by the relatively greater development of the frontal brain lobes in Man, unconscious of the fact that also in the ontogenesis of the Apes there is a time when an orbital keel does not exist, though at that time the relative development of these lobes will in essentials not be different from the relations as found in the adult stage [1]). As we see, the development of the frontal brain becomes an insufficient explanation for the relative lack of a rostrum in Man. The view of BOULE and ANTHONY, however, is commonly accepted by others, as, for instance, MARTIN, GREGORY and TILNEY.

With the same right one could reverse the causes of the facts, if one could be content with such a simple causality in a living organism, saying: in Apes there is a big rostrum because the orbital roofs protrude far and high into the brain cavity, whereas in Man the orbital roofs protrude less so that the orbital keel is much less developed. Such a view, moreover, takes much better into account the ontogenetic development, in which, as we saw, there occurs a stage where the situation is probably equal in all Primates, no protruding orbital roofs existing at all. It is not

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[1]) According to BRUMMELKAMP ('40), for instance, the relative size of different field areas of the neocortex of the sheep remains practically constant during ontogenesis.

very probable that from these stages on the relative development of the frontal brain will be very different between the genera [1]).

For these reasons it is incorrect to call a brain itself primitive, because it has a somewhat more than usually pronounced rostrum orbitale. For in saying so one would be inclined to consider the physiological possibilities of that brain as being primitive. In such cases it would be better to say that the morphological organisation of the organism, especially that of the facial part of the skull including the orbits, shows primitive or, rather, progressive features.

Another fact which is seldom taken into account by authors dealing with the rostrum orbitale and which is nevertheless of importance for the understanding of the shape of this formation, is the difference in position of the orbits towards each other in Man and Anthropoids. Already by the mere fact that in homotopic levels the absolute distance between the orbits (septum interorbitale of SCHWALBE) is greater in Man, the human orbital keel seems less high than it really is. This is to be seen in figure 5.

The process of rostral development may be described as follows. In Anthropoids as in Man in certain embryonic stages the orbits are situated quite laterally to the nose cavity, so that the orbital roofs and the tectum nasi are practically in one plane. Afterwards the roofs become vaulted primarily by a relatively greater increase of the orbital height, so that they begin to protrude into the frontal part of the brain cavity. In this way the space between the roofs in which the orbital keel of the frontal brain is situated develops. In all Primates this process will be the same. Then, in the group of Summoprimates, we see in non-hominids the beginning of another process: the turning upwards of the orbits as a whole. This implies that the line which connects the centres of the orbits, lying at first beneath the level of the lamina cribrosa, rises above this level. The later process is not equally pronounced in all Anthropoids. Most constant and most expressed it is in Gorilla, whose skull among all Summoprimates shows the greatest morphological differences between younger and older stages (see also Fig. 6). In the adult Gorilla the frontal part of the orbits lies beside the big sinus frontalis instead of beside the nose cavity as in Man. The ontogenetic ascent of the orbits has been already investigated by BOLK.

This ascent of the orbit as a whole cannot be observed in Man, where in this respect fetal conditions are preserved. More or less the same thing was already pointed out in 1885 by LISSAUER, who wrote that the splanchnocranium in Anthropoids turns upwards during ontogenesis, whereas that in Man turns downwards. KEITH, who also investigated these relations, says very pithily: "The face of the adult covers what was the forehead of the infant Anthropoid." This ontogenetic difference also distinctly appears

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[1]) BRUMMELKAMP ('38) found that the relation between the surface area of the frontal brain and that of the whole neocortex is constant in all Primates, Prosimiae excepted. Thus there is no elective development of the frontal lobes within the group of the higher Primates.

from the figures given by him in his article of 1910. Like BOLK, KEITH concludes that in many features "the infantile condition of Anthropoids becomes the permanent condition of adult Man."
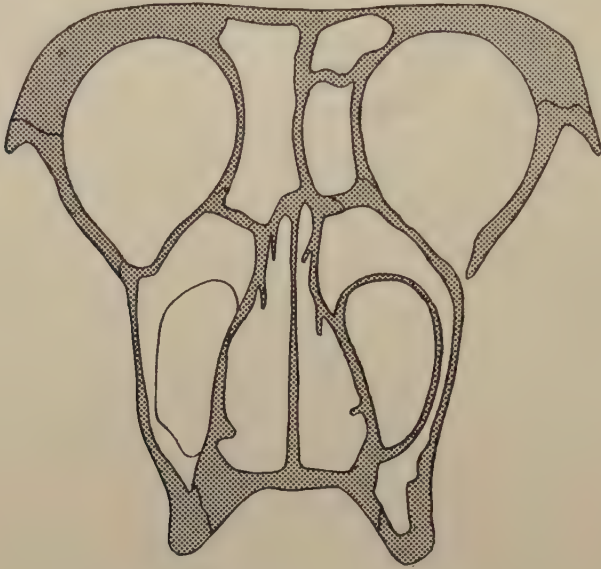

Fig. 6. Frontal section through skull of adult Gorilla. (After BOLK.)

When with these facts in mind we study the endocranial casts of Anthropoids, fossil and recent Men, given in figure 7, we see that they may play a part here also. In the photographs a constantly smaller rostrum is to be found from Ape to recent Man, which is, and not for the least part, also due to a constantly growing distance between the orbital cavities. Likewise a gradually decreasing penetration of the facial part of the skull into the neurocranium is to be deduced. In the cranial casts of lower palaeolithic Man such as the one from Ngandong, the rostrum orbitale is still much more pronounced than in recent Man. From lower palaeolithic Man onwards conditions are nearly the same as in his recent descendant. It is often forgotten that also in these relations a considerable individual variability exists, so that one has to be careful in giving a certain rostrum the epitheton primitive, especially since it is rarely expressed in measurements.

The fact, pointed out by C. U. ARIËNS KAPPERS, that in Anthropoids and also in some fossil Men the upper border of the rostrum, viewed from aside, runs upwards in relation to his lateral subcerebral horizontal line (see Gibraltar cast), whereas in recent Man this border remains beneath that horizontal, is without any doubt due to the fact that in Apes, and also in a much slighter degree in some fossil human skulls, the roof of the orbit, causing that border, runs more upwards in a frontal direction relative to his horizontal line than in recent Man. This again is due to the fact that in the more "primitive" cases the orbits and therefore also their superior walls are relatively more turned upwards in the skull. This agrees with our statement that in Anthropoids and also, though in a minor degree, in lower palaeolithic Man, as we could observe from

some unpublished diagrams, the whole orbital pyramid points more upwards in a frontal direction than in recent human skulls.

The further forward we go in the skull of the large Apes, the more the orbits are situated above and beside the nose cavity, instead of lying only beside it. In a strict sense, therefore, it is impossible to make any direct deductions about the breadth of the inner nose in Anthropoids or about the development of the organ of smell with the aid of the interorbital breadth index, as SCHWALBE did, because the place of measurement of the breadth of the nose cavity on this level lies far above the roof of the latter.

This is very apparent from our figure 6, a frontal section of a Gorilla skull after BOLK, sawn just behind the third molar, so even somewhat caudally to the level, where the breadth of the "septum interorbitale", between the dacrya, must be measured. In Anthropoids there are no parts of the nose lying between the dacrya, as is the case in Man (see fig. 2), where not only the upper part of the nose cavity but also the sinus ethmoidales are situated between those two points. In the large Apes the sinus frontalis, if present, lies between them. Figure 6 again clearly demonstrates the high position which the orbits have relatively to the nose cavity in Anthropoids.

Concluding we can say that, simply speaking, the development of a rostrum orbitale is determined by the ontogenetic protruding of the orbital roofs into the frontal part of the brain cavity and that the extent of this protrusion is a different one among the group of Summoprimates, being greatest in Gorilla and least in Man. Therefore the development of a rostrum is primarily due to a more or less progressive development of the facial sphere. This being fetalised or conservatively developed in Man, the reduced rostrum may be explained as a secondary fetalised feature. The development of the frontal brain in the sense of BOULE and ANTHONY will scarcely play a part, if any, here.

Though we think we have demonstrated that the greater or lesser development of a rostrum is primarily due to a greater or lesser protruding of the orbital roofs into the brain cavity, we do not mean to say that these roofs exert any direct pressure on the frontal brain lobes, thus forming the rostrum. By no means this protruding is the simple direct causal factor in rostral development. As everywhere in an organism the diverse structures are adjusted to each other in a way which is of a far more complex nature. At best one could say that, comparing the facial sphere and the brain, the facial sphere in Anthropoids plays an increasingly important part during ontogenesis, whereas in Man the balance, as existing in young fetal times, is preserved.

Acknowledgements.

assistance in making the röntgenograms of which one is published in this paper and to Prof. C. U. ARIËNS KAPPERS for the photograms of the endocranial casts of his collection.

Summary.

1. The topographic relations between the orbits and the nose cavity in Man and Anthropoids have been studied during ontogenesis. Special attention has been given to the relations between the orbital roofs and the cribriform plate, forming together the relief of the fossa cranii anterior.

2. It has been found that in young embryonic stages of all Summo-primates the anterior part of the skull base is almost flat. The orbits are situated beside the nose cavity. In consequence no U- or V-shaped valley between the orbital roofs, in which the rostrum orbitale of the frontal brain in adult stages lies, occurs as yet. Neither in embryonic Apes nor in embryonic Man is there such a rostrum at all.

3. Some factors are concerned with the formation of this valley in later ontogenesis. Primarily, and this is true of all Summoprimates, there is an increase of the vertical orbital height relative to that of the nose cavity. This causes a protrusion of the orbital roofs into the brain cavity. In the second place the orbits of Anthropoids turn upwards in the face so that their position, being at first quite lateral to the nose cavity, changes to one above and beside it. Therefore the orbital roofs will protrude relatively more in Anthropoids than in Man. These processes are almost completed at a young infantile stage.

4. The difference in form between the rostrum in Man and in the Apes is due to at least the following factors:

a. The absolutely lesser distance between the orbits in Anthropoids which makes the rostrum much narrower.

b. The greater height of the valley in which the rostrum lies in relation to the total height of the frontal brain in Anthropoids.

c. The turning upwards of the orbits in Anthropoids.

The greatest absolute height of the rostrum in Man is not very different from that in Anthropoids and therefore the whole difference, apart from the greater narrowness in Apes, is merely a relative one.

5. In pleistocene Man conditions are intermediate between recent Man and the Anthropoids, although they are much nearer to modern human conditions. Some more progressive, anthropoidal, developmental features, however, are apparent.

6. Certainly the relatively lesser development of the rostrum in Man is not simply due to a greater development of the frontal lobes, as was thought before. The cause may be sought in the more conservative development of the topographic orbital relations in Man in contrast to the more progressive one, especially that of the facial sphere, in Anthropoids. The relations in the orbital sphere of Man are fetalised in the sense of BOLK's theory.
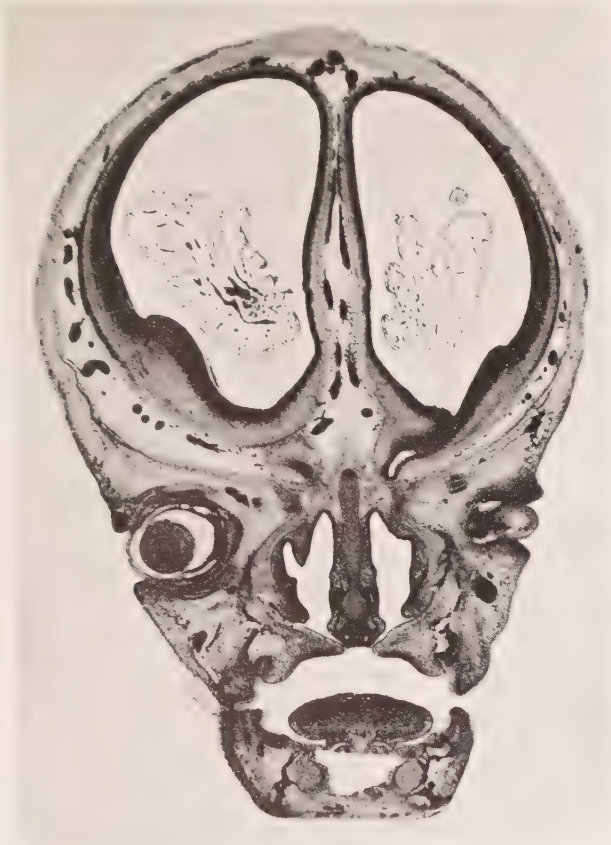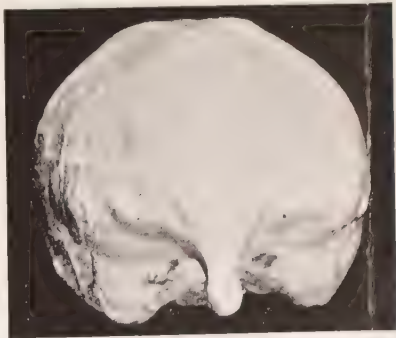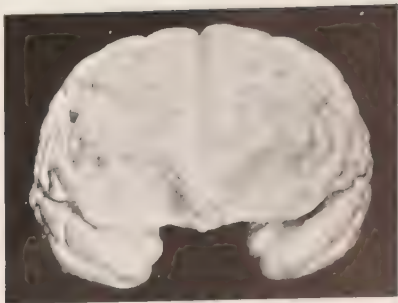
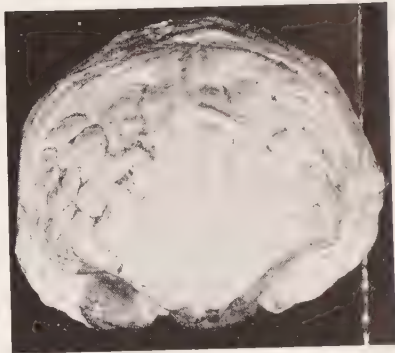Fig. 1. Frontal section through head of human embryo aged 2 months.
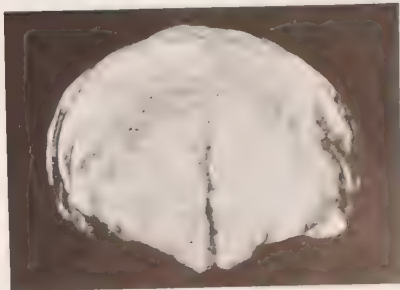

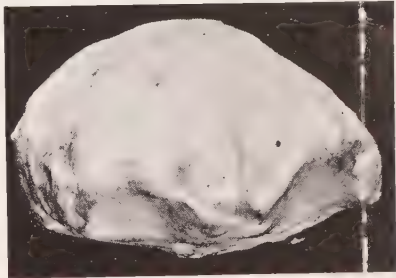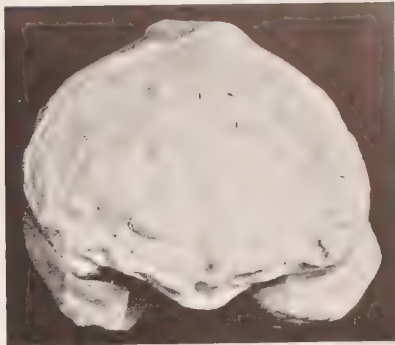Fig. 4. X-ray photogram through head of Gorilla fetus.
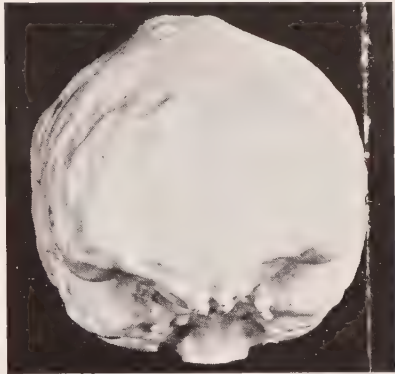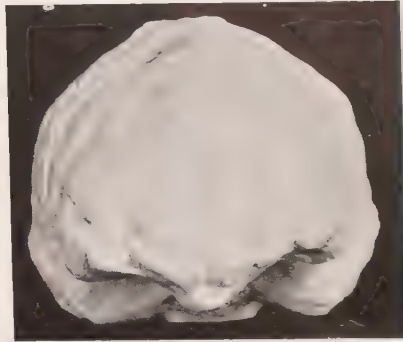
Gorilla.

Chimpanzee

Sinanthropus.

Ngandong V.

Gibraltar (from aside).

Predmost III.

Torres straits.

Eskimo.

Fig. 7. Photograms of endocranial casts from Anthropoids, fossil and recent Man in different orientations.

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(Communicated at the meeting of October 26, 1940.)

First Part.

I. *The maternal oxygen- and carbon dioxide dissociation curves during pregnancy.*

In earlier work (1937) it had been established that the influence of pregnancy on the oxygen dissociation curve in the cow differs from that observed by Barcroft et al. (1934) in the goat in this respect, that in the cow the curve remains within the limits of the field which we had fixed for normal non-pregnant animals, whereas in the goat the curve leaves the normal field at a gestation of 10 weeks, remaining outside during all the further time of pregnancy.

It was in the last two months only that in the cow the lower part of the curve, going until an oxygen pressure of 20 mm maximally, had been shown to become more inflected, so that the right limit of the field was slightly crossed by the curve.

It is true that this field is considerably wider than that in the goat, and also exceeds the dimensions determined for any other animal examined, man included.

The course of the oxygen dissociation curve had been watched in one and the same animal, using 3 cows and starting from the 4th month of pregnancy.

It seemed desirable to us to follow this course of the curves in a greater number of animals and using a different technique, because of the importance of that lower part of the curve for its shape and position, and for its steepness, which had been found to be increased during the last months. The position of the points for low oxygen pressure had not satisfied us and incited to new measurements. At the same time the carbon dioxide dissociation curves (carbon dioxide d.c.s.) had to be determined.

It was our intention to check the limits of the field for the curves of normal non-pregnant animals meanwhile.

Four healthy adult Frisian cows were used for these measurements. The blood was drawn from the jugular vein at various moments of pregnancy as is shown by table 1. (For table 1 see next page).

The blood was collected in a wide glass tube which contained a 5 % solution of ammonium oxalate with $2\frac{1}{2}$ % sodium fluoride. The solution

had been brought to a pH of 7.4. It had a volume of 1/50 of that of the blood with which it was thoroughly mixed. In some experiments the anti-coagulant was used in crystalline form.

TABLE 1.

Number of animal	Duration of pregnancy in months
11	4
	6
	$6\frac{1}{2}$
	8
	9
12	2
	3
	6
	8
	$8\frac{1}{2}$
	9
13	6
	7
	$7\frac{1}{2}$
	$8\frac{1}{2}$
	9
14	5
	6
	9

However much BARCROFT's differential manometer had satisfied us, VAN SLYKE's constant volume apparatus was used in these experiments because of our interest in the position of the points for low oxygen pressure especially. The advantages offered by this apparatus, when a great number of determinations have to be carried out, was another reason to choose it.

Since a practically unlimited volume of blood was at our disposal, the determinations were made using 2 cc of blood.

The carbon dioxide curves were determined by measuring the carbon dioxide in 1 cc of true plasma, obtained from oxygenated blood, in the usual manner.

Results.

Before describing the oxygen d.c. during pregnancy, the limits for the field within which the curves for normal non-pregnant animals are found will be discussed. The shape, given in an earlier paper (1938), was verified by determining the curves in six normal non-pregnant animals. The new field, demonstrated in Fig. 1, agrees for far the most part with that given previously; its lower part has undergone a slight correction. In the same figure the field for the goat and for man has been drawn for comparison.
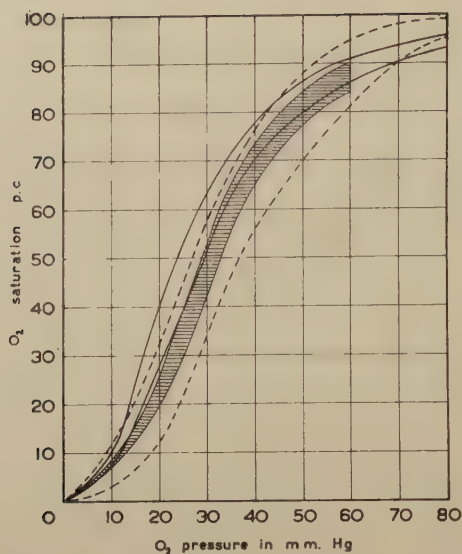
Fig. 1. The area of the O_2 dissociation curves of normal cows, not pregnant, and not having been so recently, is given by the dotted lines. CO_2 pressure 40 mm Hg., temp. 38° C. The field for the goat has been shaded, that for man has been drawn. The curves for the goat and for man according to BARCROFT.

The carbon dioxide d.c.s for true plasma of oxygenated blood, determined at the same time for the same animals, are enclosed by the limits given in Fig. 2. As the reader will find there the reserve alkali varies between 47.5 and 57.5.
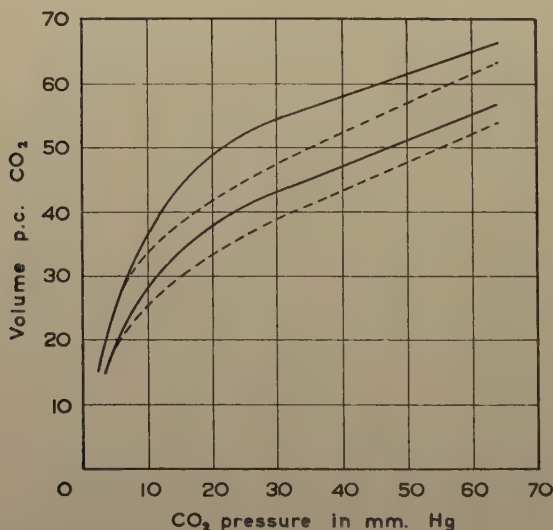
Fig. 2. The area of the CO_2 dissociation curves for true plasma of oxygenated blood of normal cows, not pregnant and not having been so recently, is given by the drawn lines. The dotted field is that for oxygenated blood, as had been determined in previous work. Temp. 38° C.

It is not sure that Fig. 2 gives a fully correct image of the difference between plasma and blood. The state of nutrition of the animals and the kind of nourishment may be supposed to have its influence here, since the true plasma curves were obtained from animals which had passed the summer in the meadow whereas the first set of curves, one excepted, had been determined in the beginning of spring, after the animals had been kept in the stable during the winter. In Fig. 3 a comparison is given of the carbon dioxide d.c.s of whole blood and true plasma from one and the same animal.
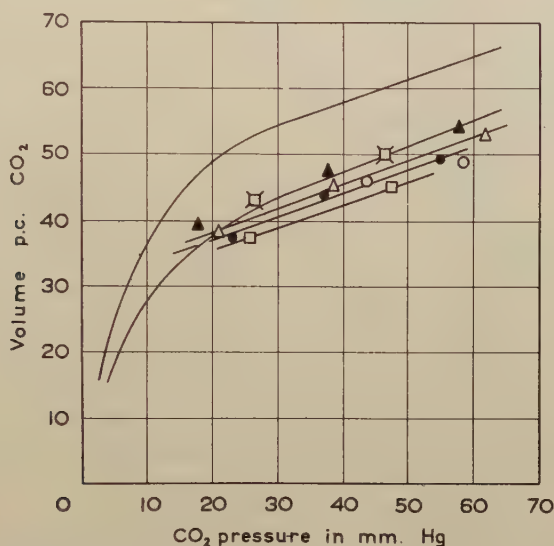

Fig. 3. Carbon dioxide dissociation curves of (a) maternal oxygenated blood (□) and oxygenated true plasma (⋈). The animal was pregnant during 8 months. (b) maternal oxygenated blood (○) and oxygenated true plasma (△) of an animal, pregnant during 7 months, together with those of the foetal blood (● and ▲). Temp. 38° C.

Coming now to the oxygen d.c.s of the pregnant animal, it was established that the influence of gestation is perceivable in two respects. (1) The curves become steeper during pregnancy and (2) they become more inflected during the last 6 months. Speaking in general the position of the curves remains within the limits of the normal field. In some instances the right border line of the latter may be surpassed slightly by the lower end of the curve as a result of the increased inflection observed here during the 7th, 8th and 9th month.

The results, obtained in earlier experiments (1937) with respect to position and shape of the curves have been confirmed. The more steeple position however was found not to be restricted to the last months only, but it was detected to occur earlier, starting from about the 4th month of pregnancy.

In Fig. 4, 5, 6 and 7 a set of oxygen d.c.s is given, together with the carbon dioxide d.c.s, which may be considered as representative.
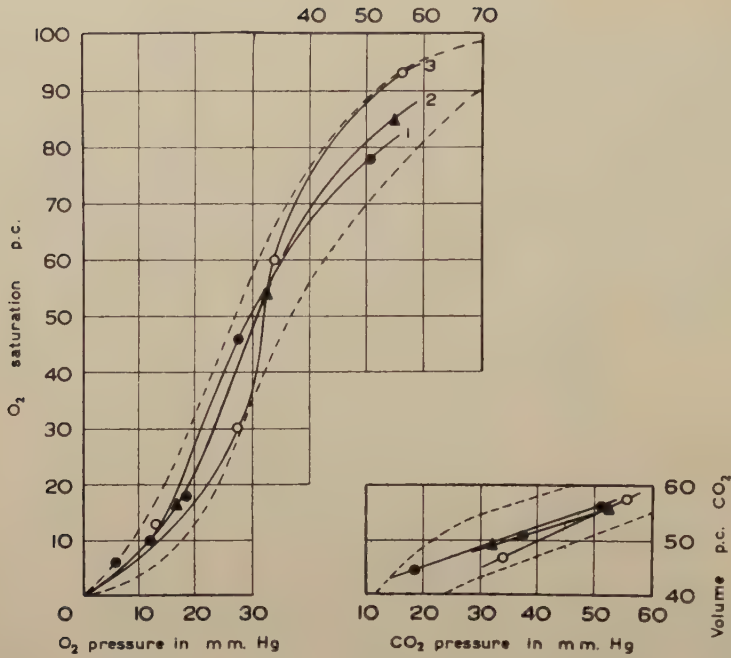
Fig. 4. O_2 dissociation curves and CO_2 dissociation curves of cow 12, pregnant about 2 months (1), 6 months (2) and 8 months (3). The dotted lines give the limits of the field for normal non-pregnant animals. For the O_2 d.c.s. CO_2 pressure 40—43 mm Hg; the CO_2 d.c.s. have been determined for true oxygenated plasma. Temp. for both 38° C.
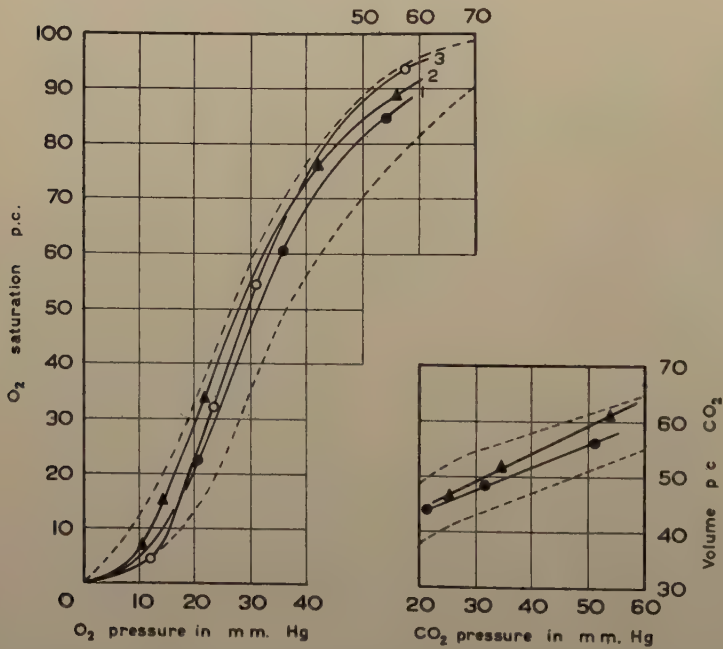

Fig. 5. Cow 13, pregnant 7 months (1) and 9 months (2). The third curve (3) was obtained one week later, 3 hours before parturition. The dotted lines give the limits of the field for normal non-pregnant animals. Rest as in Fig. 4.
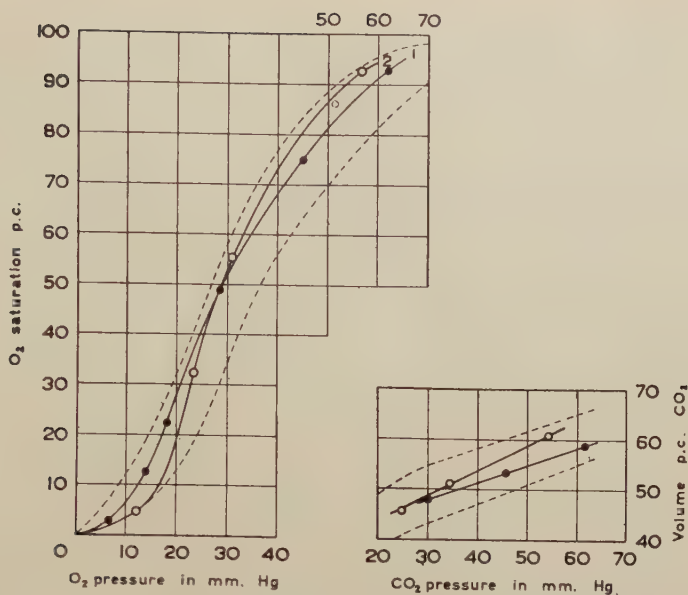
Fig. 6.   Cow 13, pregnant 8½ months (1). Curve (2) was obtained from blood, taken 3 hours ante partum. The dotted lines give the limits of the field for normal non-pregnant animals. Rest as in Fig. 4.
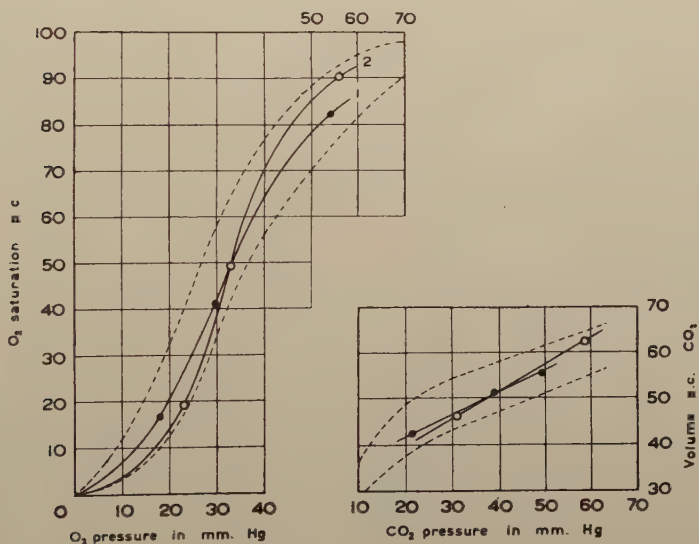

Fig. 7.   Cow 14, pregnant 5 months (1) and 6 months (2). The dotted lines give the limits of the field for normal non-pregnant animals. Rest as in Fig. 4.

So it must be concluded that although the changes of the maternal oxygen d.c. during the period of pregnancy may not be called salient, they nevertheless are distinct when the bearing of the curves is watched in one and the same animal.

Concerning the causes by which these changes are produced the only

conclusion that can be drawn with security is this that they are not found in changes of hydrogen-ion concentration.

Firstly it would be difficult to understand how both changes described could be produced by changes of the pH. Moreover the reserve alkali was rather constant during the greater part of the period of gestation, so that its influence was quite insufficient to produce the changes of position of the curves established.

The course of the reserve alkali in the pregnant animal is shown by Fig. 8. It shows that the carbon dioxide combining power of its blood is not lower than in the non-pregnant animal and that there is a noteworthy constancy of alkali reserve up to the ninth month. During that time it ranged between 50 and 55. In one animal (no. 13) it rose somewhat above this limit, reaching 58 in the midst of the 8th month. Then, at the end of the 9th month, a general rise is observed. In two instances in which the blood had been examined immediately before parturition and some few minutes after it, a fall of the reserve alkali could be stated to have taken place during this short time (about 1 hour). The values, found after parturition, have been plotted on the ordinate p.p.
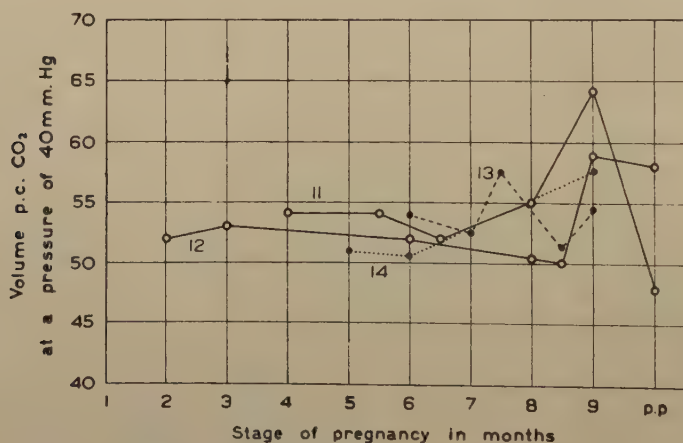

Fig. 8. Reserve alkali during the period of pregnancy in 4 pregnant cows. The figures at the curves give the number of the animals; p.p. = some few minutes post partum. The determinations were made in true plasma of oxygenated blood. Temp. 38° C.

The values of the reserve alkali and those for the hydrogen-ion exponent of true plasma at the various stages of gestation, at a carbon dioxide pressure of 40 mm Hg, are given in table 2. The latter were calculated by using the HENDERSON–HASSELBALCH formula.

In how far the changes of the oxygen d.c.s are due to alterations of the haemoglobin molecule on the one hand and to the effect of electrolytes on the other remains still an open question at this moment.

In none of the animals the maternal alkali reserve was lower than in normal non-pregnant animals. Most often it was found to correspond with the higher part of the normal field for non-pregnant animals.

TABLE 2.

Number of animal	Duration of pregnancy in months	Reserve alkali in true plasma (Vol. p.c. CO_2 bound at a CO_2 pressure of 40 mm Hg)	pH
11	4	54	7.38
	6	54	7.38
	$6\frac{1}{2}$	52	7.36
	8	55	7.38
	9	64	7.44
	10 min. p. p.	48	7.33
12	2	52	7.36
	3	$53\frac{1}{2}$	7.37
	6	52	7.36
	8	$50\frac{1}{2}$	7.35
	$8\frac{1}{2}$	50	7.35
	9	59	7.42
	5 min. p. p.	58	7.41
13	6	54	7.38
	7	$52\frac{1}{2}$	7.36
	$7\frac{1}{2}$	$58\frac{1}{2}$	7.41
	$8\frac{1}{2}$	51	7.35
	9	$54\frac{1}{2}$	7.38
14	5	51	7.35
	6	$50\frac{1}{2}$	7.35
	9	58	7.41

It is remarkable that so fundamental differences, as observed here, can be stated to exist at various points between ox- blood on one hand and the blood of man, and even the blood of the goat, on the other, both animals approaching each other so nearly in many respects. In the goat the maternal curve moves in the opposite direction and at the half of gestation it falls outside the normal field where it remains during the further time of pregnancy. BARCROFT et al. (1934) found this displacement to be accounted for by the change in reaction: the increased hydrogen-ion concentration. In the ox there is no displacement to the right side at all and the change of hydrogen-ion concentration is quite different.

Also in man the results obtained in studying the maternal oxygen d.c. differ from those in the ox; EASTMAN, GEILING and DE LAWDER (1933) who examined the curve of the mother immediately after birth of the child, found it distinctly displaced to the right. A corresponding result has been obtained in 1936 by LEIBSON, LIKHNITZKY and SAX.

Consequently the gap between maternal and foetal oxygen d.c. which

is found in the ox- blood as well as in the other species mentioned, is effected in the cow by removal of the foetal curve exclusively.

II. *The dissociation curves of the foetal and the maternal blood.*

Some years ago we examined the relation between foetal and maternal dissociation curve in the cow and published the curves, obtained from an animal, 8 months pregnant (1938). The opportunity to continue the work in a greater number of animals and at various stages of pregnancy was wanting at that moment.

In this chapter the results obtained in a greater number of experiments will be given.

Method.

Both oxygen- and carbon dioxide d.c.s were determined. The maternal blood was taken from a carotid artery, a sample of foetal blood being drawn from one of the umbilical vessels, after Caesarian section had been done under local anaesthesia; the foetus remained connected to its mother under conditions as physiological as possible. The blood was collected in the way as described in the foregoing pages.

The moment of pregnancy at which the curves were determined are clear from the following table.

Age of foetus in months	Sex	Number of cases
3	m	1
$3\frac{1}{2}$	m	1
4	f	1
$5\frac{1}{2}$	f	1
7	f	2
7	m	1
8	f	1
8	m	2
$8\frac{1}{2}$	—	1

Results.

In all cases irrespective of the age of the foetus, the oxygen dissociation curve of the foetal blood was found to the left of the maternal. The difference in position between both was considerable and the foetal curve was found to lie outside the field for normal non-pregnant animals in 9 out

of 11 cases. In the two remaining instances the curves were found either for the greater part or completely within the limits of the normal field.

As was described in the first chapter of this paper the maternal curves remained within the normal field. In some instances the inflection in the last month of pregnancy had increased in such a way that the lower part of the curve crossed the right limit of the field but this was an exception. Of course the curves obtained at various ages of the foetus, can hardly be compared, since they were obtained from different animals.

In the figures 9—13 a set of curves is given for both mother and foetus on various points of foetal life.
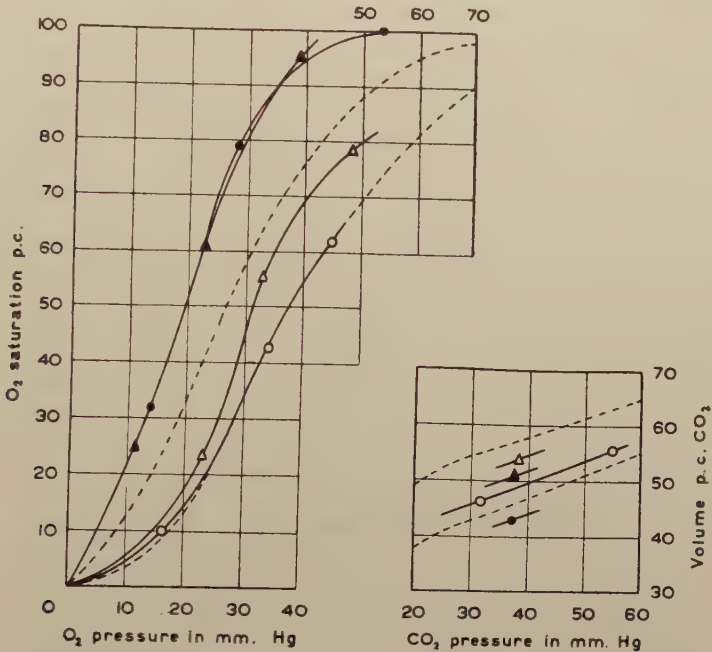

Fig. 9. O₂ dissociation curves of two foetus, old 3 (●) and 3½ months (▲) respectively, together with those of the mothers ((○) and (△)). The foetal curves coincide for a great part. CO₂ pressure 42—44 mm Hg., temp. 38° C. At the bottom the corresponding CO₂ dissociation curves for true plasma of oxygenated blood. The field for the normal non-pregnant animals has been dotted.

In all figures given here the foetal curve is found outside the normal adult field, as it was the rule. Two other examples are given by Fig. 14 in which the curve for a foetus, 4 months old, is found to have its position for the greater part within the normal area, whilst the dissociation curve of a foetus of 8¼ months is found entirely within that field. If we consider the carbon dioxide d.c.s of Fig. 14 the explanation of these exceptions does not offer difficulties. In both foetus the reserve alkali was low and remained below the normal field. In the foetus of 8½ months it was even extremely low and less than 50 p.c. of the value which may be considered
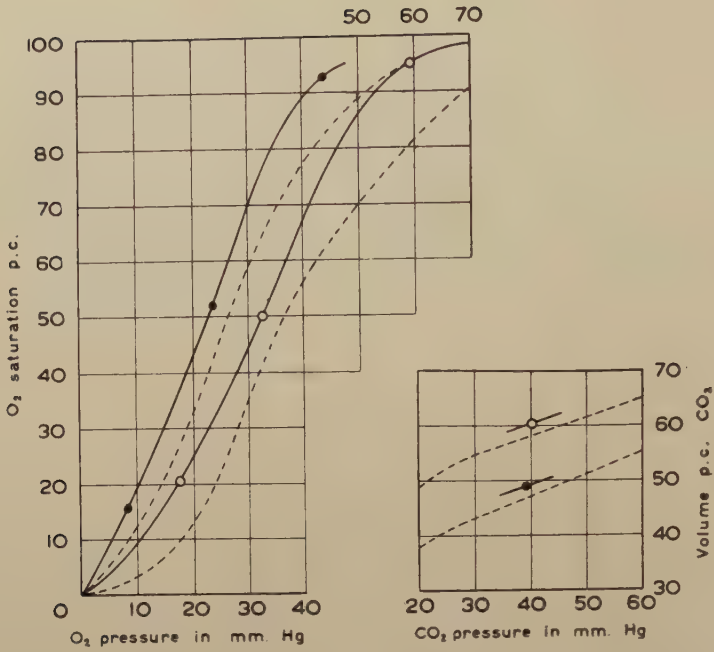
Fig. 10. Age of the foetus 5½ months. The rest as in Fig. 9.
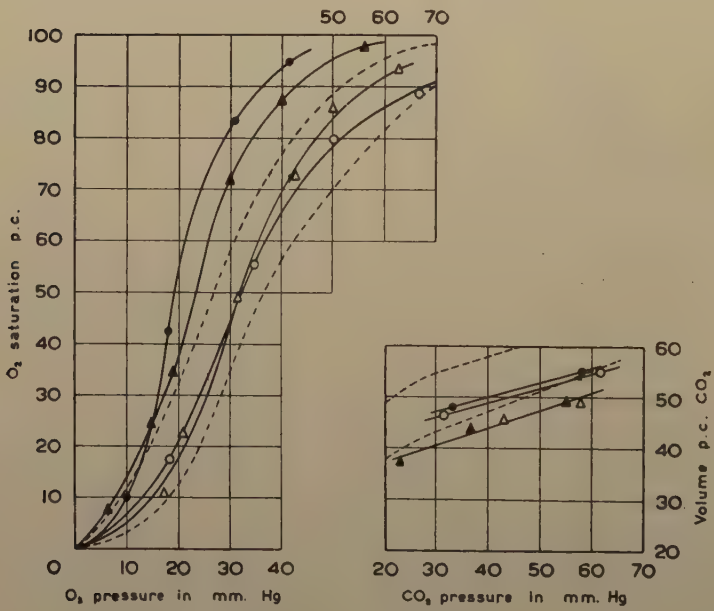
Fig. 11. Age of both foetus 7 months. The rest as in Fig. 9.

Fig. 12. Age of foetus 7 (●) and 8 (▲) months respectively. The rest as in Fig. 9.
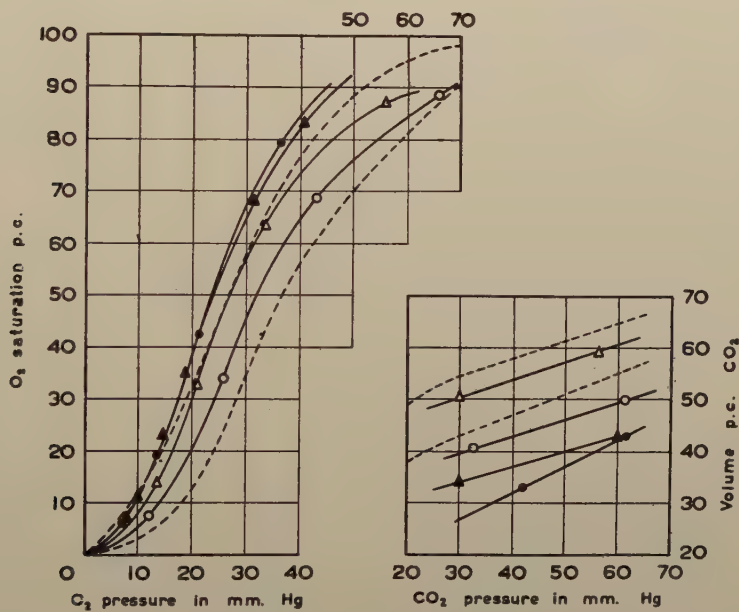
Fig. 13. Age of both foetus 8 months. The rest as in Fig. 9.

as being normal. It is the Bohr effect by which the position of the curves of Fig. 14 is brought about.

How is the course of the reserve alkali during foetal development? During the first 6 months of intra-uterine life the foetal carbon dioxide

curves fall within the field, established for normal non-pregnant animals, be it in its lower part. In the seventh months' foetus the curve was found a little below that field. As intra-uterine life procedes the gap between the two increases (See Fig. 12 and 13). This had also been found in the foetus of 8 months, described in an earlier paper (1938).
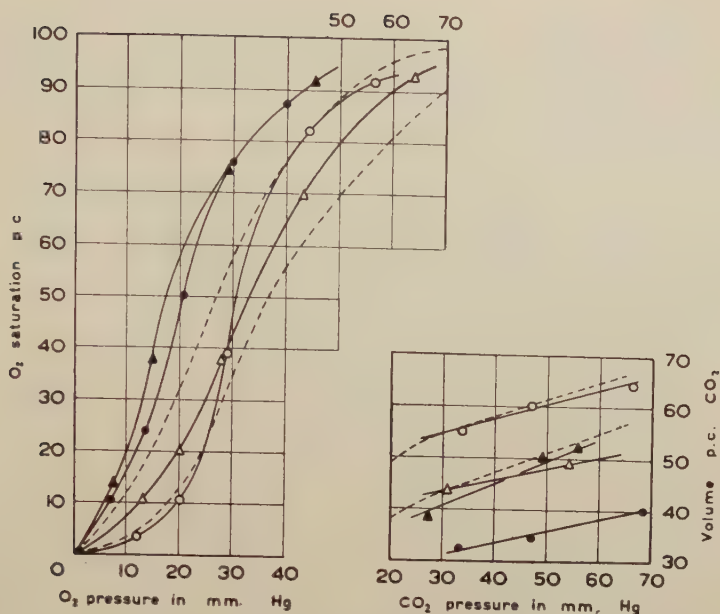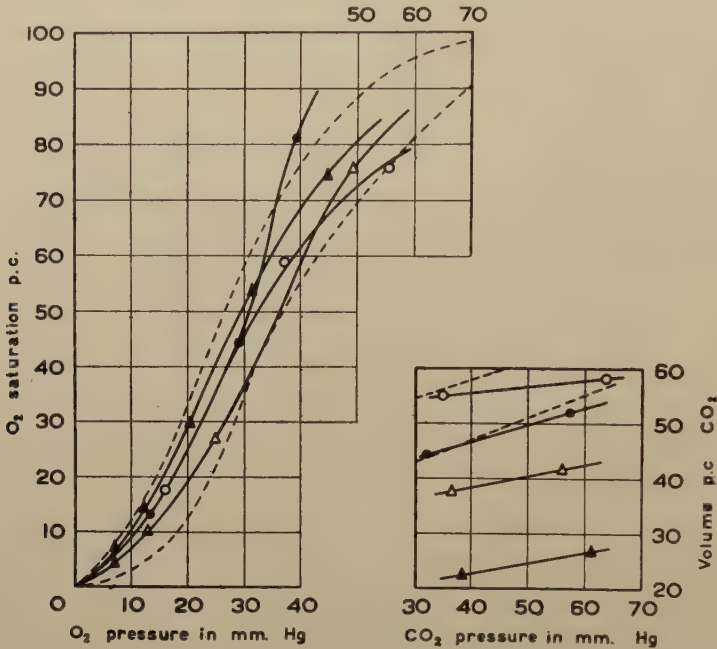
Fig. 14. Age of foetus 4 (●) and 8½ (▲) months respectively. The rest as in Fig. 9.

In the mother cows the reserve alkali was rather high also in this series. Up to about the 7th month of pregnancy the position of the curves correspond even to the higher half part of the normal field for non-pregnant animals. Later on a rather great dispersion of results was observed, which must probably be attributed to the condition of nutrition of the animals of this series. The pregnant animals described in chapter (I) were all fully sound and in an excellent state of nutrition. They belonged to the permanent stock of cattle of one of the Veterinary Institutes of the Faculty. The animals however which were sacrificed to the experiment together with their foetus had been bought for this purpose. Their state of nutrition was considerably less. So the lower reserve alkali, found in some of them will probably find its explanation in this circumstance. In 8 out of 11 experiments maternal reserve alkali was considerably higher than the foetal, in the remaining 3 cases there was scarcely any difference between the two.

A review of the reserve alkali, measured in both mother and foetus, is given in Fig. 15. Besides the 11 animals treated in this chapter, other

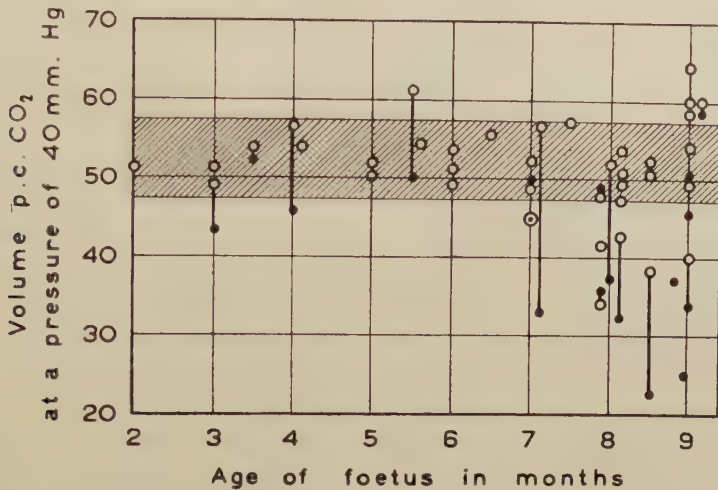measurements made in pregnant and in new-born animals, have been collected.

Fig. 15. Reserve alkali in maternal blood (○) and in blood of the foetus at various ages (●). The CO_2 dissociation field for normal non-pregnant adult animals has been shaded. The points indicate the volume p.c. CO_2, bound by true plasma at a gas pressure of 40 mm Hg.

The figure shows the relation between mother and foetus and moreover the fall of foetal reserve alkali in the last months.

Returning to the oxygen d.c. another point of difference between maternal and foetal curve has still to be stated, viz. concerning the shape of the curve. The foetal curve is less inflected than the maternal and during the first six months of prenatal life especially its lower part is so stretched that the hyperbolic form is approached. Until the 7th month this shape is conspicuous as may be seen in the figures 9, 10 and 14. So the moment of change in the foetal oxygen d.c. obviously coincides with the fall of alkali reserve.

Discussion.

This coincidence may be of importance for the foetus from a physiological point of view since both changes cooperate in facilitating giving off oxygen in the area of low oxygen pressure, as will be found in the foetal tissues. Especially the stretched oxygen d.c. may be called favourable. It is not only the oxygen binding power which, at a low oxygen pressure, is thus promoted but the volume of oxygen given off will be relative great at a slight fall of oxygen pressure. So the respiratory needs of foetal tissues during the first months may be considered as being contented in a very satisfactory way in consequence of this hyperbolic curve.

As the young embryo grows the oxygen needs of its tissues, the muscles especially, will increase considerably and the process of gas change will

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be performed at a higher level of the curve, i.e. at a higher oxygen pressure. The more inflected curve, exhibited by the foetal blood starting from the 7th month may be considered to meet this demand in a striking way. In the area of higher oxygen pressure oxygen will be given off now more readily than it was allowed by the hyperbolic curve.

A set of oxygen dissociation curves has been given for the goat by BARCROFT et al. (1934). These authors also found a gap between maternal and foetal curves but it was produced by displacement of both maternal and foetal curve. Whereas the foetal curve was found to the left of the normal, starting from about the 12th week, the maternal was drifting to the right and had already left the normal area at the 10th week of gestation.

The carbon dioxide balance between maternal and foetal blood in the goat was examined by KEYS (1934). His results differ considerably from ours, observed in the ox. Contrary to our findings he found a markedly higher alkali reserve in the foetal than in the respective maternal blood. The average foetal blood was able to bind 22 p.c. more carbon dioxide than the corresponding maternal at the same tension. The foetal reserve alkali was not much lower than what he considered as normal for the goat. The maternal one however was considerably below it. Unfortunately KEYS did not measure the reserve alkali in true plasma of either oxygenated nor reduced blood but in whole blood. If we try to make his figures comparable with ours the result is that the reserve alkali in true plasma of oxygenated blood was here between 35 and 40 in 4 cases and about 45 in the fourth.

It is difficult to follow his conclusions concerning the course of the reserve alkali during foetal life since they are based on determinations, partly made by others and the impression is given that they were made under conditions which were not very favourable.

Some figures for the reserve alkali in the pregnant woman are found in literature. The conclusion is that there is a definite drop of reserve alkali as pregnancy proceeds. WILLIAMSON (1923) found a minimum (33 in multiparas, $43\frac{1}{2}$ in primiparas) at the end of gestation. After parturition it rose so that on the 10th day after delivery a normal level was reached again. In the child at term the carbon dioxide binding power of the blood was considerably higher than in the maternal blood.

So there are points of difference in many respects between the goat and the ox.

1. In the goat the first sign of pregnancy is exhibited by the maternal curve which, according to the authors as the result of increased hydrogen-ion concentration, leaves the normal field between the 7th and 10th week whilst any shift in the foetal curve is still lacking. In the cow there is no fall of reserve alkali in the mother, the maternal oxygen dissociation curve does not move to the right but even slightly to the left, maintaining however its position within the normal field.

2. In the cow the inflection of the curve increases.

3. The foetal curve however is much more mobile in the cow than in

the goat. As soon as in the 3rd month, i.e. in the first third part of intra-uterine life, it has taken its position far from the normal field (Fig. 9), whilst in the goat it is not before the 13th week (i.e. at the end of the second third part of gestation) that its removal becomes distinct and the curve partly falls outside the limits of the normal area.

4. In the goat the higher part of the curve remains inside those limits up to the 19th week.

5. Further the authors mentioned found the foetal and the maternal curves drawing towards one another again in the last week of foetal life of the goat, and one week later, after birth, the foetal curve had returned again in the adult field, both maternal and foetal curve having the normal degree of inflection. In the cow the foetal curve persists considerably longer in its foetal state concerning both position and shape. In the new-born calf the curve, although nearer to the maternal than it was in the foetal time, is still outside the normal field (1937).

So the divergences, exhibited by the foetus, are far more intensive in the cow than in the goat, and apparently the young animal is born in a condition, differing much more from the adult than the kid of the goat does.

6. It is interesting that in the goat the foetal blood, though not more alkaline than the normal, is more alkaline than the maternal. The same is the case in man. The reverse relation is found in the cow.

Although in the cow the difference between maternal and foetal curve is brought about by the properties of the foetal oxygen dissociation curve exclusively, that difference and its importance for foetal respiration is not

TABLE 3.

Oxygen pressure required in order to give an oxygen saturation of 50 p.c.

Age of foetus	Goat (BARCROFT)			Age of foetus	Cow		
	Foetus	Mother	Difference		Foetus	Mother	Difference
10 weeks	32	36	4	3 months	20	37	17
11 ,,	30	36	6	3½ ,, (3)	18	34	16
13½ ,,	24	34	10	4 ,,	30½	32½	2
14 ,,	30	38	8	5½ ,, (2)	23	33	10
15 ,,	24½	37	12½	6 ,,	10	30	20
16½ ,,	22	35	13	7 ,, (3)	20	32	12
18 ,,	25	40	15	8 ,, (4)	20	32	12
19½ ,,	21	35	14	8½ ,,	29	36½	7½
20½ ,,	24	34	10				
At birth	30	36	6	At birth (6)	22½	31½	9

The figures between brackets give the number of observations from which the average value has been taken.

less considerable. Also in the cow the avidity for oxygen is far greater in the foetal than in the maternal blood, and at a given oxygen pressure foetal blood will be saturated to a notably higher degree than the maternal. This will be evident from the figures, collected in table 3, in which the respective figures observed in the goat by BARCROFT, are given for comparison. The table shows the gas tensions required in order to give a 50 p.c. saturation to the blood. For both mother and foetus, the pressure required is lower in the cow than in the goat at a corresponding moment of pregnancy. The difference however between mother and foetus is generally greater in the cow than in the goat.

In table 4 the figures are given for the reserve alkali of true plasma for both mothers and foetus. The pH has been calculated.

TABLE 4.

Reserve alkali in maternal and foetal true plasma.

Age of foetus in months	Sex	Reserve alkali		Corresponding pH (calculated)	
		mother	foetus	mother	foetus
3	m	49½	44	7.34	7.28
3½	m	54½	52	7.39	7.36
4	f	56	46	7.40	7.30
5½	f	60½	50	7.43	7.34
7	f	49	49½	7.33	7.34
7	f	44	44	7.28	7.28
7	m	57½	33½	7.40	7.15
8	f	43	32	7.28	7.15
8	m	54½	37½	7.39	7.20
8	m	47	47	7.30	7.30
8½	—	39	23	7.22	6.98

In man, in which the oxygen dissociation curve for mother and child was examined immediately after birth of the baby (LEIBSON, LIKHNITZKY and SAX (1936)) the same events have been found as in the goat, in so far that the maternal curve is displaced considerably to the right, whilst the foetal curve corresponds nearly with that of the non-pregnant adult.

This displacement of the maternal curve in man and in the animals, except for the ox, deserves attention because of its physiological consequences. It shows a greater ability of the blood to give off its oxygen and foetal respiration will thus be improved, provided that maternal blood will be oxygenated in the lungs to the same degree in spite of the higher oxygen pressure in the blood. Is it reasonable to assume that in general

alveolar oxygen pressure will be high enough to do so? No doubt it is an interesting point to investigate the composition of alveolar air in the course of pregnancy. It may be remembered here, that during pregnancy we found an increase of oxygen content of the alveolar air in the dog (1937). Starting from the 4th week of pregnancy, i.e. before the midst of foetal life, alveolar oxygen pressure increased with 6 to 12 p.c., whilst at the same time carbon dioxide pressure had fallen to under 90 p.c. of its original value.

In the description of our first observation, two years ago, it was already emphasized that differences of reserve alkali could not be responsible for the difference between maternal and foetal oxygen d.c. The new experiments affirm this conclusion since the pH of the foetal blood either agreed with that of the maternal or even remained below it. In spite of this fact the foetal oxygen curve was found at the left, which feature proves that the displacement to the right, promoted by the altered hydrogen-ion concentration, is counteracted by another influence by which it is inverted to a displacement in the opposite direction. Moreover it is not the position only but also the shape, by which both curves differ.

It is very probable that the signs observed are effected by the properties of foetal haemoglobin, which differ from those of the maternal. For the chemical work done in this field by many workers, the reader may be referred to a report given for the Sixteenth International Physiological Congress (1938). The question in how far electrolytes play a part here, cannot be settled.

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Psychology. — *Die menschlichen Kommunikationsformen und die sog. Tiersprache*. I. Von G. RÉVÉSZ. (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of October 26, 1940.)

1. *Die Kommunikation im Allgemeinen.*

Wollen wir den Begriff „Kommunikation" dem Sprachgebrauch entsprechend verwenden, so müssen wir die Bezeichnung „Kommunikation" auf solche Verbindungen beschränken, wo es sich um wechselseitige, von einander abhängige Veränderungen im Verhalten der kommunizierenden Individuen bezw. Gruppen handelt. Demzufolge können wir die *Kommunikation* als ein erbbiologisch fundiertes und zu den allgemeinen Lebensäusserungen und Lebensbedürfnissen gehörendes, auf Gegenseitigkeit beruhendes Verhalten zwischen artgleichen und artungleichen lebenden Wesen definieren. Die Kommunikation ist also mit der zwischen Individuen und Gruppen bestehenden gegenseitigen Verbindung nur dann eindeutig charakterisiert, wenn die Verbindung für die Art beförderlich und das Mittel der Verbindung, des Aufeinanderwirkens ausgebildet und der psychophysischen Beschaffenheit der miteinander in Kommunikation tretenden Arten bezw. sozialen Gruppen angepasst ist.

Aus dieser erklärenden Definition folgt, dass *die vitale Bedeutung, die Gegenseitigkeit, ferner die Fixierung und Angepasstheit der Mittel die wesentlichsten Merkmale jeder Form der Kommunikation bilden*. Wo diese Merkmale fehlen, wird man von Kommunikation nicht sprechen dürfen, sonst müssten wir alle Reaktionen lebender Wesen, insofern sie durch ein anderes Wesen ausgelöst werden, als kommunikative Aeusserungen ansehen. In diesem Falle müssten u.a. die an sich zweckmässigen, aber einseitigen Reaktionen, wie etwa die Flucht der Tiere beim Bröllen des Löwen, der Schreckruf der Hühner beim Anblick der Schlange, die Angriffsstellung der Katzen beim Riechen von Mäusen, das Weinen des Kindes beim Wahrnehmen fremder Gesichter als Folge einer kommunikativen Verbindung zwischen Beutetier und Löwe, Huhn und Schlange, Katze und Maus, Kind und Mensch betrachtet werden. Bei allen diesen Fällen fehlt das der Kommunikation zugrunde liegende oder erst durch sie zustande gebrachte Zusammenspiel der Partner.

Besonders muss man sich hüten, sich von der blossen Tatsache des Zusammenseins, der Vergesellschaftung irreführen zu lassen, und sie selbst schon als Ausdruck der Kommunikation zu betrachten. Nichts wäre unrichtiger als diese Auffassung. Die soziale Verbindung stellt noch kein zuverlässliches Zeichen für das Vorhandensein einer Kommunikation dar. Es gibt Tiere, die gelegentlich oder ständig zusammenleben, einander

aufsuchen, Schlaf-, Schwarm- und Wandergesellschaften bilden, gemein-
sam nach Nahrung suchen und Feinde angreifen, ohne dass zwischen ihnen
Kommunikation in dem angegebenen Sinne bestehen würde. Dass Glieder
einer solchen Gemeinschaft sich keines Kommunikationsmittels bedienen,
verhindert das Zusammenleben, das Bestehen gemeinschaftlicher Inte-
ressen, selbst eine differenzierte Arbeitsteilung nicht. Wie vielseitig diese
sozialen Tätigkeiten auch sein mögen, sie verdanken ihre Existenz unver-
änderlichen biologischen Gesetzen, die die Glieder der Gemeinschaft zu
gewissen vorausbestimmten Verhaltungen geradezu zwingen.

Was die Ausdrucksform der Kommunikation anbelangt, so muss man
wissen, dass nicht alle Laute, Bewegungen, Haltungen, die andere Wesen
zu bestimmten zweckmässigen Reaktionen veranlassen, ohne Weiteres als
Kommunikationsmittel zu gelten haben. Wenn der Schmerzlaut eines
Tieres seine Artgenossen zur Flucht veranlasst, so folgt daraus noch nicht,
dass diese Reaktion — die nicht einmal zweckmässig zu sein braucht —
auf wechselseitiger Beziehung beruht. Gelegentlich kann ein beliebiger
Reiz bei derselben Tierart dieselbe Reaktion auslösen. Auch der mensch-
liche Schmerzlaut bildet an sich kein Kommunikationsmittel zwischen den
Menschen, trotzdem kann er als Anzeichen eines bemitleideten Zustandes
uns zur Hülfeleistung veranlassen. Allerdings kann der Mensch einen
Schmerzlaut mit der Absicht ausstossen, um mit seinen Artgenossen in
Kontakt zu treten. Wird dieser beabsichtigte Laut von den anderen als
solcher verstanden, so ist die Lautäusserung zum Kommunikationsmittel
ad hoc geworden. Seine Kommunikationsfunktion liegt aber nicht im Laut
selbst, sondern in der Absicht.

Die gegebene Definition gibt genügende Anhaltspunkte, um zu ent-
scheiden, ob einer reaktiven Handlung eine kommunikative Tendenz zu
Grunde liegt oder nicht. Damit ist aber der Weg zur Erforschung des
Kommunikationsproblems noch nicht gesichert. Wir müssen versuchen zuver-
lässige Gesichtspunkte zu finden, die uns in diesem äusserst komplizierten
Gebiet zu leiten vermögen. Es scheint mir, dass diese Gesichtspunkte durch
Feststellung der *Kommunikationsformen* zu gewinnen sind. Diese klassifi-
katorische Arbeit ist umso dringlicher, da sie zu Problemstellungen führt,
die sowohl für die Entwicklungspsychologie, als auch für die Sprach-
psychologie von grundlegender Bedeutung sein dürften.

a) Die erste Unterscheidung, die wir treffen müssen, ist die zwischen
absichtlicher und *unabsichtlicher* Kommunikation. Absichtliche Kommuni-
kation lässt sich nur bei Menschen antreffen. Sie ist mit der Willens- und
Sprachfunktion in engster Beziehung [1]). Unabsichtliche, zwangsläufige
Kommunikation zeigt sich vornehmlich bei Tieren, gelegentlich auch bei
Menschen, insbesondere bei kleinen Kindern. Die Mittel der absichtlichen
Kommunikation bilden die Gebärden- und Lautsprachen, Schrift- und Bild-

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[1]) Siehe darüber ausführlicher im zweiten Teil dieser Arbeit: Die Tiersprache. Proc.
Ned. Akad. v. Wetensch., Amsterdam, **43**, N⁰. 10 (1940).

sprachen, schliesslich die konventionellen Signalsysteme. Die Mittel der unabsichtlichen Kommunikation bestehen aus Lauten, Bewegungen, Verhaltungen, die dem Triebmechanismus angehören.

b) Die Kommunikation kann entweder dem Interesse des „Senders" oder des „Empfangers" dienen oder zu Gunsten beider. Die ausschliesslich dem Lebensinteresse des Senders dienende Verhaltungsweise können wir *autistische*, die nur auf den Empfänger bezugnehmende *altruistische*, schliesslich die wechselseitige Interesse fördernde Verhaltungsweise *soziale* nennen. Ausschliesslich im Interesse des „angesprochenen" Partners geäusserte Zeichengabe kommt nur bei Menschen vor; sie ist immer von Absicht getragen. Meiner Kenntnis nach gibt es kein Tier, das Laut- oder Bewegungszeichen ausschliesslich zum Wohl anderer Tierindividuen hervorbringen würde; der Sender ist dabei immer beteiligt. Der Ruf der Henne zur Ernährung, Reinigung und Pflege ihrer Brut ist nicht ohne jeden Gewinn für die Mutter. In der Fürsorge liegt die Befriedigung wichtiger Lebensbedürfnisse der Mutter eingeschlossen: die Kleinen leisten aktiv zugleich als nehmende und als gebende Mitglieder des Gynopaediums der Mutter Dienste. Demgegenüber sind die autistischen und besonders die sozialen Kommunikationen in der menschlichen Gesellschaft wie im ganzen Tierreich weit verbreitet.

c) Die wichtigste Unterscheidung bezieht sich einerseits auf *gerichtete, adressierte*, andererseits auf *nicht-gerichtete, richtungslose* Kommunikationen. Diese Unterscheidung hat besonders vom entwicklungspsychologischen Standpunkt aus Bedeutung.

Die gerichtete Kommunikation setzt eine interindividuelle Beziehung voraus; es geht aus ihr die Tendenz hervor, den Empfänger des Rufes zu einer bestimmten Handlung zu veranlassen. Die ungerichtete Kommunikation unterscheidet sich von der gerichteten dadurch, dass bei ihr gerade diese *Tendenz* fehlt, durch gewisse Verhaltungen Einfluss auf bestimmte Individuen oder Gruppen auszuüben und zwischen Sender und Empfänger eine interindividuelle Beziehung, eine Art von Partnerschaft zu schaffen. Diese richtungslose Form der Verbindung kann dennoch rechtmässig als Kommunikation bezeichnet werden, da sie geeignet ist, durch der Art angepasste Mittel Tierindividuen bezw. Tiergruppen zu zweckmässigen und artfördernden Handlungen zu veranlassen.

Als Beispiele von gerichteten, adressierten Kommunikationen kommen in erster Linie die menschliche Sprache in ihren verschiedenen Erscheinungsformen (Laut-, Gebärden- und Zeichensprache), ferner einige zweckdienliche Laut- und Bewegungsäusserungen des kleinen Kindes in der vorsprachlichen Entwicklungsphase in Betracht [1]).

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[1]) KARL BÜHLER hat in seiner „Sprachtheorie" (1934) die Sprache durch drei weitgehend unabhängige Variabeln charakterisiert, nämlich durch *Ausdruck*, *Appell* und *Darstellung*. (In seinen früheren sprachpsychologischen Arbeiten verwendete er die Termini: Kundgabe, Auslösung und Darstellung). Diese Merkmale sind besonders geeignet die Sprachsituation des Menschen darzustellen, dagegen nicht für eine Beschreibung der

Obgleich die an bestimmte Individuen bezw. Gruppen gerichteten kommunikativen Aeusserungen vornehmlich für die soziale Verbindung der Menschen bezeichnend sind, finden wir sie gelegentlich auch im Tierreich. Bestimmte Lautäusserungen und Körperhaltungen der Tiere zwecks Anlockung der Geschlechter, wozu die mannigfaltigen Stimmäusserungen und Balzstellungen der Vögel gehören, die Lockrufe des Muttertieres, ferner eine Anzahl von Lauten, Bewegungen und Körperhaltungen domestizierter Tiere und Affen dienen als Beispiele. Hierher gehören auch der durch Dressur und Gewohnheit entstandene Kontakt zwischen Mensch und Tier [1]).

Allerdings ist es nicht immer leicht zu entscheiden, ob eine tierische Laut- oder Bewegungsäusserung gerichtet oder ungerichtet ist, ob eine tierische Reaktion als Folge einer gerichteten oder einer ungerichteten Aeusserung anzusehen ist. So wird sich z.B. in den meisten Fällen der sog. Warnungsruf, durch den eine ganze Herde zur Flucht veranlasst wird, als ungerichtete Kommunikationsform deuten lassen. In jedem konkreten Fall muss die Frage gestellt werden, ob die Lautäusserung wirklich einen Warnungsruf, ein Appell darstellt, um die Genossen von der Gefahr abzuwenden und sie zur Flucht zu drängen, oder geht es nur um einen nicht-gerichteten, nicht-adressierten Laut, etwa um einen Schrecklaut, der die Herde instinktiv zur Fluchtreaktion treibt. Meiner Ansicht nach hat die letztere Deutung eine grössere Wahrscheinlichkeit. Das Tier adressiert den Ruf nicht an seine Artgenossen, um ihnen gleichsam kundzugeben, dass eine Gefahr im Anzuge ist, sondern der Ruf stellt einen blossen Ausdruck seines *eigenen* Furchtzustandes dar, der bei den übrigen Mitgliedern der Herde triebhaft Schreck und demzufolge Fluchtreaktion auslöst. Welchen Eindruck auch die Verhaltungsweise des „signalierenden" Tieres und der Herde auf uns machen möge, es scheint kein zwingender Grund vorzuliegen, diese und ähnliche Fälle zu den adressierten Kommunikationen zu rechnen. Hat man die Gelegenheit einmal das ganze Schauspiel einer durch Schreckruf in Bewegung gesetzten Herde zu beobachten, so findet man es äusserst unwahrscheinlich, dass das lautgebende Tier

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Kommunikation im allgemeinen. Erstens schliesst der Begriff Appell bereits die Aufforderung zu einer Handlung in sich, was jedenfalls für die nicht-gerichteten Kommunikationsäusserungen nicht gilt, ausserdem hat dieses Wort einen zu imperativen Charakter (Befehl-Handlung). Zweitens bezieht sich das Moment der Darstellung nur auf die Sprache und nicht auf andere kommunikative Modalitäten. Schliesslich beschränkt sich das Schema BÜHLERs auf den Sender, die Reaktion des Empfängers wird dabei nicht berücksichtigt, wodurch die Gegenseitigkeit der kommunikativen Verbindung nicht recht zum Ausdruck kommt. Da BÜHLER sich nur die Aufgabe stellte, die Relationsfundamente des Sprachereignisses festzulegen, und nicht, wie ich, die Erscheinungsformen der Kommunikation im allgemeinen darzustellen, kann sein Schema für die Klassifikation der Kommunikationsformen nicht in Betracht kommen.

[1]) Die adressierte Kommunikation schliesst die Absichtlichkeit nicht notwendig in sich. Schon wegen des Fehlens der Sprachfunktion können wir bei Tieren nicht annehmen, dass die adressierten Laut- und Bewegungsäusserungen mit bewusster Zielvorstellung und vorstellungsmässiger Antizipation des Zieles verbunden sind.

die Absicht hätte, seine Genossen vor der Gefahr zu warnen. Sein unge-
stümes, rücksichtloses Benehmen während der Flucht weist nicht auf eine
Partnerschaft, auf eine interindividuelle Beziehung.

Die sogenannte Fühlersprache der Ameisen, die vor allem dazu dient,
den Speisebrei abzugeben, oder das Ausstossen eines kurzen Tones der
wachehaltenden Bienen, der die Genossen des Stockes zum Ansammeln
veranlasst, sind auch Fälle, die ich nicht zu den gerichteten Kommuni-
kationsäusserungen rechnen möchte. Selbst der sog. Werbetanz der von
einem ergiebigen Futterplatz zurückkehrenden Bienen, der die andern zum
Ausflug nach dieser Stelle aufreizt, scheint nicht adressiert zu sein, obwohl
die ausgezeichnete Beschreibung dieses Vorganges durch v. Frisch leicht
den Eindruck erweckt, als ob die ,,tanzenden" Bienen die Tendenz hätten,
ihre Genossen von einem entdeckten Futterplatz zu verständigen [1]).

Es liegt nicht in meiner Absicht die Möglichkeit des Warnrufes bei
Tieren in Abrede zu stellen. Ich stehe nur auf dem Standpunkt, dass solange
keine unabweisbaren Argumente für adressierte, spezifisch gerichtete Laut-
oder Bewegungsäusserungen vorliegen, man bei der Interpretation solcher
Fällen die grösste Vorsicht walten lassen soll.

Die Frage, ob in einem konkreten Fall gerichtete Kommunikation vorliegt
oder nicht, lässt sich am ehesten entscheiden, wenn man prüft, ob zwischen
dem Sender und Empfänger ein *Kontakt* angenommen werden kann. Mit
dem Wort ,,Kontakt" wollen wir die Wechselseitigkeit und die von *beiden*
Teilen ausgehende Tendenz (Absicht) des Zusammenwirkens, des Auf-
einander-Gerichtet-Seins, des Eingestellt-Seins bezeichnen. Mit Rücksicht
auf die besonderen Entstehungsbedingungen und Auswirkungen der kom-
munikativen Aeusserungen wollen wir also vom Kontakt erst dann sprechen,
wenn zwischen Individuen und Gruppen auf Grund einer *bereits vorhan-
denen, in mannigfaltiger Weise in Erscheinung tretenden wechselseitigen
Beziehung mit Hilfe von zweckmässigen und auf die Art abgestimmten
und von ihnen ,,verstandenen" Mitteln ein Zusammenspiel* zustande kommt.

Kontakt entsteht in erster Linie zwischen Individuen und Gemeinschaften
von gleicher Art; aber auch zwischen artverschiedenen Wesen lässt sich
gelegentlich diese Beziehung beobachten. Die Artungleichheit an sich
schliesst das Entstehen des gegenseitigen Kontaktes keineswegs aus. Es ist
nicht so selten, dass zwischen Haustieren, z.B. zwischen Hund und Katze
kameradschaftliche Beziehung entsteht, die alle wesentlichen Merkmale des
interindividuellen Kontaktes aufweist.

Die Frage, zwischen welchen gleichartigen oder ungleichartigen Indivi-
duen Kontakt zustande kommen kann, lässt sich nicht von vornherein
beantworten. Besteht aber einmal ein Kontakt, so ist anzunehmen, dass
dieser sich bei zahlreichen Lebenssituationen äussern wird. Weisen also
Tiere nur eine einzige, bezw. eine sehr geringe Anzahl Kommunikations-
äusserungen auf, dann müssen triftige Grüde vorliegen, um diese Kommu-

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[1]) K. v. Frisch, Die Sprache der Bienen. 1923.

nikation trotzdem als gerichtet zu betrachten. Es ist nämlich sehr unwahr-
scheinlich, dass eine so tiefgreifende wechselseitige Beziehung, wie die des
Kontaktes, in der Lebensweise der Tiere nur ganz spärlich in Wirkung
treten sollte. Auch von diesem Standpunkte aus wird man z.B. die sog.
Fühlersprache der Ameisen und die Lautäusserungen der Hunde nicht zu
den adressierten Kommunikationen rechnen.

Der Kontakt wurzelt nicht immer im Biologischen. Beispiele liefern
dafür die geistig fundierten Verständigungsformen der Menschen. Zu-
weilen lassen sich aus der Sphäre des rein-Biologischen herausragenden
Kontaktbeziehungen auch bei Tieren feststellen. So ist z.B. der Kontakt
zwischen gewissen Haustieren und Menschen nur zum Teil Resultat der
Domestizierung, der Hauptsache nach Ergebnis der individuellen Gewohn-
heitsbildung und Erfahrung.

Ich lege ein ganz besonderes Gewicht auf das *Kontaktmoment*. Kommu-
nikative Verbindungen mit und ohne Kontakt stellen die grössten Diffe-
renzen dar, die im Gebiet der Kommunikation überhaupt anzutreffen sind.
Nur bei dieser Unterscheidung lassen sich die Formen der Kommunikation
hinsichtlich ihrer Bedeutung für das Einzelindividuum, die ganze Verhal-
tungsweise der kommunizierenden Sozietätsmitglieder und die Möglichkeit
einer Kommunikation zwischen Mensch und Tier verständlich machen.

2. *Die Sprache als spezifisch anthropologischer Begriff.*

Die angeführten Beispiele haben uns gelehrt, dass Kommunikationen
zwischen artgleichen und artverschiedenen Individuen und Gruppen in
verschiedener Weise entstehen können, nämlich gerichtet wie ungerichtet,
absichtlich wie unabsichtlich, aktiv wie reaktiv, instinktiv wie geistig fun-
diert. Sie treten in verschiedenen Formen zum Ausdruck, als Bewegung,
Körperhaltung, Lautäusserung, Gebärde und Sprache. Es ist deutlich
geworden, das Kommunikation die Sprache grundsätzlich nicht voraus-
setzt, was sich schon daraus ergibt, dass es Kommunikationsformen
gibt, die unabhängig von der Sprache entstehen und sich entwickeln. Die
Sprachfunktion ist auch eine der Kommunikationsformen, selbst die
höchstentwickelte, aber dann eine Kontaktform sui generis, mit besonderer
Struktur und besonderen Funktionen und Ausdrucksmitteln ausgestattet.

Auf die morphologische und funktionelle Verschiedenheit der Sprache
von den übrigen Kommunikationsformen muss besonders hingewiesen
werden, da in der zoologischen und tierpsychologischen Literatur gelegent-
lich die Tendenz wahrzunehmen ist, die Sprache *als die allgemeine Kom-
munikationsform* zu betrachten, was zu Folge hat, dass man alle Laut- und
Bewegungsäusserungen der Tiere, durch welche sie auf ihre Artgenossen
und andere Tiere einwirken, unter dem Begriff der „Sprache" unterzu-
bringen versucht[1]. Das kann so weit gehen, dass selbst die Verhaltungs-
von der Sprache der *Tiere* spricht und diese von der der *Menschen* unter-

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[1] F. HEMPELMANN, Tierpsychologie, 1926, S. 529.

weisen bei der Paarung der Geschlechter als Formen der Sprache gedeutet werden.

Die Verwendung des Sprachbegriffes für Bezeichnung aller jener instinktiven Aktionen und Reaktionen, die den gegenseitigen Verkehr fördern und den gemeinsamen Interessen dienen, ist nicht nur unzweckmässig, sondern auch unstatthaft. Daran ändert nicht, wenn man nachdrücklich scheidet. Denn werden einmal die tierischen Kommunikationsformen unter den Oberbegriff „Sprache" gebracht, so besteht die Gefahr, zwischen tierischen und menschlichen Kommunikationsformen weitgehende Analogien zu statuieren und spezifische Funktionen der Sprache in die tierischen Kontaktformen hineinzuinterpretieren. Der Umstand also, dass ein Forscher ausdrücklich betont, dass die „Tiersprache" zur Unterscheidung der menschlichen Sprache den logischen Aufbau, die grammatikale Struktur, die bewusste Mitteilungsabsicht usf. entbehrt, genügt bei weitem nicht, um falschen Schlüsse vorzubeugen. Nicht selten begegnen wir Forschern, die an einer Stelle die menschliche Sprache von der sogenannten Tiersprache prinzipiell unterscheiden, an einer anderen Stelle ohne jede Zurückhaltung von einer „ausdrucksvollen und kräftigen Sprache des Männchens" und von einer „Laut- und Gebärdensprache bei Anlockung der Geschlechter" sprechen.

Die Anwendung spezifisch anthropologischer Begriffe überhaupt, wie z.B. die der Sprache, des Verstandes, des Willens in der Tierpsychologie ist nicht so unbedenklich, wie es uns auf dem ersten Blick erscheinen mag. Die Gefahren sind keineswegs dadurch aufgehoben, dass man sich dessen bewusst ist, dass der Begriff im übertragenen Sinne, metaphorisch verwendet wird. Man realisiert meistens nicht, dass man bei Anwendung von Lehnbegriffen leicht der Täuschung unterliegt, als ob zwischen dem Inhalt des ursprünglichen Begriffes und dem des Lehnbegriffes weitgehende Uebereinstimmungen bestünden. Die Macht der Wortidentität kann uns leicht zur Annahme der sachlichen Identität verleiten [1]).

Eine sprachpsychologische Grundtatsache ist es, dass jeder eindeutig definierte Begriff oder jedes im Sprachgebrauch eindeutig verwendete Wort jedem Versuch widerstrebt, seinen Umfang und Inhalt nach Bedarf zu modifizieren. Daraus folgt, dass ein Wort seine ursprüngliche Bedeutung auch dann durchsetzt, wenn es mit vollem Bewusstsein im bildlichen, im übertragenen Sinne verwendet wird. Trotz besten Vorhabens tritt die ursprüngliche Bedeutung in Bereitschaft, verschmilzt mit dem neuen Inhalt und beeinflusst unbemerkt unsere Ueberlegungen und Schlussfolgerungen.

Wie verderblich die Einführung des Begriffes der Sprache in die Tierpsychologie sein kann, lässt sich nicht besser demonstrieren als durch Hinweis auf Fragestellungen, die gerade dadurch entstanden sind, dass man aus dem Begriff „Tiersprache" solche Konsequenzen zog, die nur aus

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[1]) Vergl. dazu die Ausführungen in meiner Arbeit: Gibt es einen Hörraum? in den Acta Psychologica, 1937, Band III, S. 137.

dem ursprünglichen Begriff der Sprache folgen. Eine ganze Literatur entstand, die sich mit vollem Ernst zur Aufgabe stellte, die „Sprache" der Tiere zu untersuchen, mit dem Zwecke, diese zur menschlichen Sprache in Beziehung zu bringen. Dabei sind die Forscher nicht bei den autochthonen tierischen Kommunikationsformen, bei den spezifischen Laut- und Bewegungsäusserungen der Tiere stehen geblieben, — was wenigstens darauf hingewiesen hätte, dass sie das Wort „Sprache" tatsächlich in einer ganz anderen Bedeutung verwenden wie die Sprachwissenschaft. Sie haben sich durch die Macht des Wortes so sehr beeinflussen lassen, dass sie gewissen Tieren das Verständnis der menschlichen Lautworte zuerkannten, ja sogar die prinzipielle Möglichkeit des Anlernens der menschlichen Sprache — wenn auch in sehr beschränktem Masse — ernstlich erwogen. Immer wieder wurden Beobachtungen, Erfahrungen, experimentelle Ergebnisse mitgeteilt, die in mehr oder weniger verdeckter Form die Ansicht zum Ausdruck brachten, dass zwischen den tierischen und menschlichen Kommunikationsformen ansehnliche Uebereinstimmungen bestehen und dass der Unterschied zwischen Mensch und Tier in dieser Beziehung nicht so einschneidend ist, wie das von philosophischer, teilweise auch von psychologischer Seite aus dargelegt wird.

Meiner Ansicht nach sind auch die Hypothesen über die vermeintlichen Ansätze und Uebergangsformen der Sprache [1]), die angeblich in den Warn- und Lockrufen, in Lautimitationen u.s.w. ihren Niederschlag finden, Folgen der unberechtigten Anwendung des Begriffes „Sprache". In dieser Weise werden gleichsam durch eine Zauberformel Tätigkeiten, die an sich mit dem Sprachakt nichts zu tun haben, mit der Sprache eng verbunden und geben zu irrtümlichen Lehren Anlass.

Nach diesen Auseinandersetzungen wird es niemanden überraschen, dass die Ansichten über die Kommunikationen in der Tierwelt äusserst schwankend und unklar sind. Bei der Uneinigkeit und Undeutlichkeit der Ansichten und wegen der dabei entstandenen Begriffsverwirrung schien es mir nicht überflüssig zu sein, auf diese Fragen einzugehen, und durch Klärung der Probleme das Fundament für eine befriedigende Lösung zu legen.

3. *Das Wesen der Sprache.*

Die erste und wichtigste Frage, über die wir uns in diesem Zusammenhang entscheiden müssen, ist des leitenden *Gesichtspunktes* und der *Methode*.

Zunächst müssen wir uns darüber klar werden, dass die Frage nach der sog. Tiersprache ausschliesslich auf Grund tierpsychologischen Tatsachen nicht gelöst werden kann. Jeder, der unvoreingenommen die seitens der Tier- und Entwicklungspsychologen aufgestellten Thesen und Theorien einer kritischen Betrachtung unterzieht, muss schliesslich zu der Ueber-

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[1]) G. Révész, Het probleem van den oorsprong der taal. Nederl. Tijdschr. v. Psychologie, VIII, 1940.

zeugung kommen, dass die aufgeworfene Frage durch Aufzeigen der ver-
schiedenen tierischen Kommunikationsformen und durch Hinweise auf
gewisse Dressurleistungen, die ihrerseits die widersprechendsten Deutungen
zulassen, mit logischer Gewissheit nicht zu beantworten ist. Man muss dem-
nach trachten einen logisch rechtmässigen Ausgangspunkt zu finden, von
dem die Erfahrungstatsachen eine natürliche und sinnvolle Deutung finden
können. Dieser Ausgangspunkt ist in der *Begriffsbestimmung der Sprache*
zu finden. Von da müssen wir ausgehen, selbst auf die Gefahr hin, dass
unsere theoretischen Ueberlegungen vielleicht nicht durchwegs den Erfolg
haben werden, die gegenteiligen Ansichten zu einer Revision zu zwingen.
Nimmt man sich aber die Mühe die sog. Tiersprache vom sprachphiloso-
phischen und sprachpsychologischen Standpunkte aus zu sehen, — meiner
Ansicht nach die einzig berechtigte Stellungsnahme, — so wird man nicht
bloss die Lehre von einer Tiersprache aufgeben, sondern zugleich die
Widersinnigkeit der Problemstellung in ihrer heutigen Fassung in ihrem
ganzen Umfang einsehen.

Der Begriff der Sprache ist dogmatisch nicht festgelegt. Die Entstehung
einer Philosophie und Psychologie der Sprache hat eine notwendige und
zugleich fruchtbare Erweiterung des Sprachbegriffes nach sich gezogen.
Daraus erklärt sich, dass bezüglich der Definition der Sprache, trotz weit-
gehender Uebereinstimmung der Auffassungen, keine Einigkeit erzielt ist.
Prinzipielle Argumente lassen sich gegen eine Erweiterung des Begriffes
nicht anführen, solange dieses Streben nicht zu Ausschaltung unerlässlicher
Merkmale der Sprache führt. Obgleich bei den verschiedenen Definitionen
der Sprache das Gewicht einmal auf das eine, dann auf das andere Merkmal
gelegt wird, ist es nicht zweifelhaft, dass es Eigenschaften gibt, die not-
wendig zu dem Inhalt einer jeden Definition der Sprache gehören müssen.
Für unseren Zweck genügt, wenn wir von einer Begriffsbestimmung aus-
gehen, in der keine der konstitutiven Eigenschaften und Funktionen der
Sprache fehlt, die aber auch keine unwesentliche in die Definition aufnimmt,
folglich nicht mehr und nicht weniger umfasst als das, was der Sinn und
die Bedeutung der Sprache notwendig fordert. Ich glaube, dass die von
mir gegebene Definition diesen Forderungen entspricht.

*Unter Sprache verstehen wir die Funktion, durch die wir mit Hilfe einer
Anzahl von gegliederten (artikulierten) und in verschiedenen Sinnverbin-
dungen auftretenden Laut- bezw. Bewegungs- oder Zeichengebilden
(Wörter im erweiterten Sinne) unsere Wahrnehmungen, Gedanken,
Urteile, Wünsche darzustellen und in der Absicht gegenseitiger Verstän-
digung anderen mitzuteilen imstande sind* [1]).

Diese Definition bezieht sich auf alle Arten der Sprache (Lautwort-, Ge-
bärden-, Zeichen- und Schriftsprache), die jede ihre eigene sinnlich-anschau-
liche Ausdrucksform und ihre eigene Struktur hat. Auch umfasst sie

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[1]) G. RÉVÉSZ, Die Sprache. Proc. Ned. Akad. v. Wetensch., Amsterdam, **43**,
N⁰. 8 (1940).

notwendig alle fundamentale Funktionen der Sprache, wie die benennende, kundgebende (oder mitteilende), signifische, grammatikalisch-logische Funktion. Demgegenüber schaltet sie alle Aeusserungsformen aus, die der in der Definition aufgenommenen Merkmale entbehren. Demzufolge kann eine Aeusserungsform, die keine gegenseitige Verständigung beabsichtigt, wie z.B. die natürlichen affektbetonten Ausdrucksbewegungen nicht als Sprache gelten. Auch die differenzierten Stimmäusserungen der Vögel und der Säugetiere, die keine Sinnverbindungen bilden, entsprechen den notwendigen Kriterien einer Sprache nicht. Dasselbe gilt von den Lauten und Bewegungen des kleinen Kindes in der praelingualen Periode, soweit sie unartikuliert sind und keine Darstellungsfunktion haben und nicht mit der Absicht gegenseitiger Verständigung erfolgen.

Man kann vier *Spracharten* unterscheiden; nämlich die Laut- und Gebärdensprache und die Schrift- und Bildsprache. Die natürlichen und autochthonen Spracharten sind die Laut- und natürliche Gebärdensprache, während die Schrift- und Bildsprache aus den beiden ersteren, vornehmlich aber aus der Wortlautsprache entstanden sind.

Das adäquateste und reichhaltigste Mittel der Ausdrucks und des geistigen Kontaktes stellt die *Lautsprache* dar; alle Grundfunktionen der Sprache sind an sie gebunden und die ganze geistige Tradition des Volkes ist in ihr eingeschlossen. — Die *Gebärdensprache* bildet eine Sprache sui generis, die unmittelbar aus der Sprachfunktion, ohne deutlich nachweisbare Mitbeteiligung der Lautsprache entstanden ist. Sie weist eigene Struktur- und Entwicklungsgesetze auf. — Die *Schriftsprache* ist indessen ein Derivat der Lautwortsprache, eine einfache Uebertragung dieser Sprachform in ein System von Zeichen, ganz ähnlich wie die *Fingersprache* in Bewegungen. Es liegt daher auf der Hand, dass die aus der Lautsprache transformierten Finger- und Schriftsprache ganz dieselbe grammatikalischen Kategorien wie die Lautsprache besitzen, und in ihrem Aufbau und ihrer Ausdrucksmöglichkeit grundsätzlich mit der Lautwortsprache übereinstimmen.

Auch die *konventionellen Gebärdensprachen,* wie die der nordamerikanischen Indianer [1]), der Napolitaner [2]) oder der Taubstummen müssen als Sprachen im eigentlichen Sinne des Wortes gelten. Sie sind zum Teil in Anlehnung an die gesprochene Sprache entstanden, zum Teil durch Uebereinkunft festgelegt und rationell weitergebildet. Sie werden im Sinne der Lautsprache und der durch Anschauung gewonnenen Vorstellungsschemata begriffen. Erwähnenswert ist, dass jede konventionelle Gebärdensprache, soweit sie ein in sich geschlossenes Sprachsystem repräsentiert, sich bis zu einem gewissen Grade aus eigener Kraft zu entwickeln vermag. Sie ist imstande gelegentlich solche Geschehnisse und Relationen zum Ausdruck zu bringen, die in der Lautsprache ihr Adäquates nicht oder noch

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[1]) G. MALLERY, Sign Linguage among North American Indians. 1881.

[2]) ANDREA DE JORIO, La mimica degli autichi investigata nel gestire napolitano. 1832.

nicht hat. (Beispiele liefert dafür die Gebärdensprache der Indianer und Araber [1])).

Eine besondere Stellung nimmt die *Bildsprache* (Piktographie) ein, da sie die lebendige Sprache nicht in ihrer morphologischen und grammatischen Struktur, sondern nur ihrem *Sinne nach* bildhaft darstellt. Die Bildsprache lehnt sich nicht, wie die Finger- und Schriftsprache der Lautsprache an, und ist in ihrer Struktur und Darstellungsweise von der gesprochenen Sprache bei weitem nicht so abhängig wie die natürliche Gebärdensprache. Die Bildsprache ist eine sprachliche Mitteilungsform eigener Art, die durch das Bildhafte unmittelbar Sinn erhält und verstanden wird. Sie wird vielfach in die lebendige Sprache übertragen, um verstanden zu werden; aber auch ohne Uebertragung lässt sie sich begreifen.

Demgegenüber stellen die *Signale*, die lautlichen Zeichengaben der Bergvölker, die Trommelsprachen der Naturvölker keine Sprachen dar. Aus dem Umstand, dass das Signal „rot" als „verbotener Zugang" oder ein bestimmter Trommelrhythmus als Aufforderung zum Kampf verstanden wird, folgt keineswegs, dass ein mehr oder weniger ausgebildetes Signalsystem der Sprache gleicht. Es handelt sich hier stets um willkürlich gewählte Zeichen, denen man nach Uebereinkunft diese oder jene Bedeutung erteilt, ganz ähnlich wie man etwa in der Physik mit σ 1/1000 Sec. oder mit π eine bestimmte Zahl bezeichnet. Der Unterschied zwischen Sprache und Signalsystem kommt besonders darin zum Ausdruck, dass die Signale dem Verlauf der in sprachliches Gewand umgesetzten Gedankengänge nicht folgen, jeglicher Struktur entbehren, starr sind und sich aus eigener Kraft nicht zu entfalten vermögen. Um verstanden zu werden, muss jedes Signal für sich bekannt sein; eine Ableitung des einen aus dem anderen ist nicht möglich. Ein Signalsystem tritt dadurch zu der Sprache in Beziehung, da es Sachverhalte verdeutlicht, die gewöhnlich in der Sprache ihren Ausdruck finden.

Eine besondere Rolle müssen wir im System der Verständigungsmittel den *Interjektionen* und *mimischen Bewegungen* zusprechen. Auch sie bilden keine Sprache, aber werden in die Sprache aufgenommen, insofern sie ausser Ausdruckswert noch symbolische Bedeutung haben. Es ist hinlänglich bekannt, dass diese Aeusserungen in bestimmten Situationen die Wörter vertreten können, aber man weiss auch, dass zu diesen Verständigungsmitteln nur derjenige greifen wird, der selber spricht, und diese Zeichen nur derjenige begreifen kann, der selber der Sprache fähig ist. Interjektionen und mimische Bewegungen sind demnach unselbständige Elemente der Laut- bezw. Gebärdensprache, die nur im Rahmen der Sprache Bedeutung haben.

Die Analyse der Sprachformen zeigt, dass nur eine einzige Sprachart existiert, die allen Anforderungen des gegenseitigen Verständnisses und

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[1] I. GOLDZIEHER, Ueber Gebärden- und Zeichensprache bei den Arabern. (Z. f. Völkerpsychol. u. Sprachwiss. Bd. 16).

Kontaktes gerecht wird, und das ist die Lautwortsprache. Ausser dieser müssen auch noch die Gebärden- und Bildsprachen als Sprachen von eigener Art und besonderer Ausdrucksfähigkeit betrachtet worden, da sie durch eine Anzahl von individuell gestalteten und in verschiedenen Sinnverbindungen wiederkehrenden „Zeichen" unsere Erlebnisse, Gedanken, Wünsche begrifflich [1]) oder bildhaft darzustellen und mitzuteilen befähigt sind.

Im Zusammenhang mit diesen Betrachtungen über die Sprache treten unvermeidlich Fragen auf, deren Lösung für unsere Auffassung betreffs der Stellung des Tieres im Kosmos und der Beziehung des Tieres zum Menschen von nicht zu unterschätzender Bedeutung ist.

Die erste dieser Fragen lautet: Gibt es Tierarten, die von Natur aus die Fähigkeit besitzen die menschliche Sprache zu verstehen und gar solche, die artikulierte Lautworte auszusprechen und in ihrer Bedeutung zu erfassen imstande sind? Anschliessend an diese Frage muss geprüft werden, ob Tiere durch Gebärden mit anderen Lebewesen einen Kontakt zustande bringen.

Ist der Nachweis erbracht, dass Tiere sowohl von der aktiven wie von der passiven Sprachtätigkeit ausgeschlossen sind, so entsteht die weitere Frage, ob sie über eigene, autochthone Verständigungsformen verfügen, und zwar über solche, die verwandtschaftliche Züge mit der menschlichen Sprache aufweisen.

Gelingt es uns nicht, die Existenz solcher autochthonen Tiersprachen wahrscheinlich zu machen, so bliebe noch die Frage zu behandeln, in welcher Weise, durch welche Mittel Tiere zueinander und zu den Menschen in Kontakt treten können.

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[1]) Es ist möglich Gebärden oder andere anschauliche Zeichen an Stelle des Lautwortes zu setzen und ihnen einen begrifflichen Inhalt zu geben. Es kommt nicht auf das äussere Phänomen an, sondern auf den Inhalt und die Bedeutung. So kann ein konventionelles Zeichen oder ein Bild einen Begriff vertreten. Das Wort „Haus" ist ein Begriff, aber ebenfalls das Zeichen: da es das Haus im allgemeinen und nicht exemplarisch darstellt. Allerdings scheint uns dieses Zeichen einen begrenzteren Umfang zu haben als das Wort „Haus", welches die Behausungen aller Art in sich fasst. (Vergl. dazu KANT's Lehre vom Schema in der Kritik der reinen Vernunft.)

Medicine. — *Biological Properties of Aethinyl-testosterone* [1]). By L. A. M. STOLTE. (From the Department of Pharmacology, University of Leiden. Director Prof. Dr. S. E. DE JONGH.) (Communicated by Prof. J. V. D. HOEVE.)

(Communicated at the meeting of October 26, 1940.)

I. *Progesterone-like properties.*

In 1938 INHOFFEN and HOHLWEG (1) prepared a substance, that may be called aethinyl-testosterone, pregneninolone or anhydro-oxyprogesterone, according to the view one has upon the structural formula. The most remarkable property of the substance is its progesterone-like activity upon the endometrium of the immature rabbit not only after subcutaneous but *also* after *oral* administration. Progesterone being orally inactive, the new substance immediately gained importance for the gynaecological praxis. In the course of 1939—1940 [2]) we investigated, how far aethinyl-testosterone can be considered to be biologically identical with progesterone. We further compared the quantities required of both substances, in order to obtain a certain effect. Aethinyl-testosterone indeed proved to possess a number of properties that may be called progesterone-like. Besides, we could show (and in view of the chemical nature of the substance, this was not so very surprising), that it bears testosterone-like properties and finally it proved unexpectedly to exert certain oestrogenic activities.

Other publications have appeared already in literature, in agreement with our findings: RUZICKA c.s. (2) and EMMENS & PARKES (3) also found a progesterone-like effect upon the endometrium of the immature rabbit.

COURRIER & JOST (4) and EMMENS & PARKES described a testosterone-like influence upon the cock's comb and upon the sex accessories in the male rat.

The former investigators could maintain the pregnancy of castrated rabbits with subcutaneous injections of aeth.-test. in high dosages. The latter authors found that aeth.-test. after subcutaneous as well as after oral administration produces oestrus in castrated rats.

We intend to describe in 2 papers 1. the progesterone-like and 2. the testosterone-like and oestrone-like properties of aeth.-test.

A few times aeth.-test. was administered in oily solution (max. 1 mgm pro 3 ccm); mostentimes it was emulgated in 1 ccm water with 200 mgm gummi arabicum. For the oral administration a stomach tube was used.

a. CLAUBERG-*test.* Our research pertaining to the relation, as reported by INHOFFEN and HOHLWEG ($^2/_3$ mgm progesterone subcutaneously ∞ 2 mgm

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aeth.-test. subc. ∞ 4 mgm aeth.-test. orally = 1 K E) was confined to only some experiments.

In as much as the small number of animals allowed any conclusions, we got the impression that in our experiments the relation was 1 : 4 : 8—10 instead of 1 : 3 : 6.

3 mgm aeth.-test. (subcut. in emulsion) caused a ++/+++ effect in one rabbit; for the same result 6—8 mgm aeth.-test. (orally in emulsion) were required. Conversely $^2/_3$ mgm progesterone subcut., already caused a +++ effect.

Administration in oily solution or in an alcohol-water suspension of aeth.-test. (as used by INHOFFEN c.s.) gave no better results.

In order to allow a comparison, the effect of 10 mgm test. propr. and of 20 mgm testosterone was investigated. The result was resp. +/++ and O/⊤. Herewith it had become improbable that the progesterone-like activity of aeth.-test. upon the endometrium is due to the testosterone-like properties (KLEIN and PARKES) (5) of this substance.

In our further experiments we based ourselves upon the following relation:

progesterone subc. : aeth.-test. subc. : aeth.-test. orally 1 : 4 : 8—10.

b. Formation of deciduomata.

The experiment was done as described by SHELESNYAK (6): 16 immature rats (ca 3 weeks of age) were first treated with 2.5 γ oestrone during 4 days daily in 2 injections of 0.1 cc oil each.

The animals then were divided over 4 groups and received resp. 0.2 mgm progesterone subcut., 1 mgm aethinyl-testosterone subcut., 2.5 mgm aeth.-test. orally and 0.25 mgm testosterone subcut., daily for 7 days in 2 dosages a day.

The 8th day after the beginning of the experiment a silk thread was drawn through one of the uterine horns in longitudinal direction. Autopsy the 12th day.

Macroscopically all hurted horns proved to be unevenly thickened over the total length; distinctly isolated tumors could not be shown.

Microscopically circumscript tumor-like structures, consisting in decidual cells, were found in the endometrium of the animals, treated with progesterone and in these only. Though in the other groups a scattered decidual reaction existed (but in the intact horns as well!), it could be shown by staining with Gieson that connective tissue was involved in the process, whereas in the real deciduomata in the progesterone animals no such tissue could be detected. Conclusively, it is possible to make deciduomata appear with the aid of crystalline progesterone in immature rats, that underwent a precursory treatment with oestrone.

Neither the 4-fold quantity of aethinyl-testosterone (subcut.), nor the 10 fold quantity of aeth.-test. (orally), nor finally the equal quantity of testosterone (subcut.) proved to be able to effect the same.

On the base of dates in literature, concerning oestrone and testosterone in combination with progesterone, this failure is possibly due to the oestrogenic properties of aethinyl-testosterone still to be described. (Oestrone + progesterone, vid. COURRIER (7), SELYE and MC KEOWN (8)). On the other hand a disturbing influence of the testosterone-like component must be considered improbable (testosterone + progesterone, BROOKSBY (9)).

c. *Influence upon the sensibility of the rabbit uterus to oxytocine.*

The influence upon the uterine motility was researched upon 3 castrated rabbits with abdominal window.

Rabbit I received 100 int. U oestrone daily during 12 days. With 0.01—0.001 Voegtlin U of Piton [1]) a distinct increase (when compared with "spontaneous") of the total duration of the uterine contractions could be obtained.

After the subcutaneous administration of daily 100 int. U of oestrone, combined with 4 mgm aeth.-test. orally, the threshold of the sensibility to Piton went up to 1—0.1 V U. A 9 days treatment with solely oestrone was followed by a decrease to 0.001 V U.

In rabbit II the threshold was 0.01—0.001 V U Piton after a treatment with 100 int. U oestrone daily during 10 days. After 100 int. U oestrone + 16 mgm aeth.-test. (subcut.) on 5 subsequent days it had become 10 V U; after a 11 days treatment with oestrone only, it went down as far as to 0.001—0.0001 V U.

The combination of the oestrone with 0.4 mgm progesterone made the threshold rise to 0.01—0.001 V U in 6 days.

In rabbit III a threshold of 0.001 V U was reached after 7 subsequent daily injections of 100 int. U oestrone. After 5 following days of 100 int. U oestrone + 10 mgm aeth.-test. (orally) the threshold increased to 1—0.1 V U. After 100 int. U oestrone during 9 days it had diminished to 0.1—0.01 V U; after 100 int. U oestrone, combined with 5 mgm aeth.-test. (subcut.) it ascended to 1—0.1 V U.

This showed, that aeth.-test., subcutaneously administered as well as orally, is able to lower the sensitiveness of the rabbit uterus to oxytocine. However, the uterine contractions make a very uncoordinate impression, when under the influence of aeth.-test. The uterus has a "spotty" appearance and the contractions don't proceed regularly over the surface of the organ, as is the case under the influence of progesterone. Furthermore it was observed that under the influence of aeth.-test. the spontaneous motility decreases only little or not at all. This too is contrarious to what has been reported in literature (VAN WOERKOM (10), REYNOLDS c.s. (11)) on progesterone.

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[1]) "Piton" (Organon) is an oxytocically active preparation manufactured from the posterior pituitary lobe.

d. Maintenance of Pregnancy after castration.

1.   Pregnant *rats* were bilaterally castrated on the 5th, 10th or 14th day after sperm had been shown in the vaginal smear. To begin with the day before castration the animals were treated with resp. 1.25 mgm progesterone, 5 mgm aeth.-test. (subcut.) and 10 mgm aeth.-test. orally, all in 2 portions daily unto the 18th—20th day, whereupon autopsy was performed on the following day in order to establish wether the fetusses were present and if so to count the number of the living and the dead ones. A control group was solely treated with gummi arabicum.

TABLE I.

Day of castration	Daily dosage			
	G. arab.	1.25 mgm Prog.	5 mgm aeth. test. subc.	10 mgm aeth. test. orally
5th	4 rats —, —, —, —	3 rats +, +, —	3 rats —, —, —	3 rats —, —, —
10th	3 rats —, —, —	5 rats ++, ++, +, —	5 rats —, —, —, —, —,	3 rats —, —, —
14th	3 rats ++, +, +	3 rats ++, ++, ++	3 rats ++, +, —	++, +, +

++   Living fetusses.
+   Dead fetusses or placent. remainders.
—   absence of any remainder.

As the table (I) shows, the effect of aeth.-test. orally or subcutaneously administered, is inferior by far to that of progesterone in view of the "maintenance" of pregnancy, at least when the substances are given in the aforementioned quantitative relation of 8 : 4 : 1. At autopsy living fetusses had been found exclusively in those animals in which castration had taken place on the 14th day; the same finding, however, in the (castrated) controls, treated with nothing but gummi arabicum.

The evaluation of the results in the groups, castrated on the 5th day, is complicated by the fact, that the demonstration of sperm in the vaginal smear is not the *absolute* proof, that pregnancy has existed. The nidation of the ovum does not take place in the rat but on the 5th day, sothat at the time of castration the diagnosis of an eventual pregnancy cannot yet be delivered.

The "placental sign", i.e. the appearance of blood in the vaginal smear — what occasionally happens on the 8th day of pregnancy already — is another sign of pregnancy. (The absence hereof, however, does not exclude pregnancy; the latter may have come to an end before the 8th day). In 2 of the animals of each group, castrated on the 5th day this sign was present, so that at least in these cases pregnancy has certainly existed.

2.   Six pregnant *rabbits* were bilaterally castrated on the 13th or 15th day. From the day before castration on, the animals received twice daily aethinyl-testosterone orally or subcut. during one week. A 7th rabbit, (also

pregnant and castrated) received 3.125 mgm *progesterone* daily. On the base of experiments reported in literature (COURRIER c.s. (12), ALLEN c.s. (13), PINCUS c.s. (14)) we expected this quantity to be amply sufficient to maintain the pregnancy of this animal. This appeared to be true: at autopsy, performed on the 7th day after castration, in both horns totally 5 living embryos were found, no dead ones, no placental remainders.

With aeth.-test., in the quantities as applied by us, we never succeeded in keeping an embryo alive. Neither the 4-fold of the active dose of progesterone, viz. daily 12.5 mgm aeth.-test. subcutaneously administered (2 rabbits), nor the 10-fold, viz. daily 31 mgm aeth.-test., orally administered (2 rabbits) proved to be able to procure the pregnancy of the castrated rabbit.

Even an increase of the dosage unto as much as 30 mgm subcut. (1 rabbit) and 80 mgm orally (1 rabbit) failed to yield any result, except placental remainders and, sometimes recognizable, fetusses.

These findings are not in disagreement with those of COURRIER's and JOST's. In their experiments 3 rabbits, castrated on the 4th day of pregnancy, received 5—10 mgm aeth.-test. orally daily; 2 other animals 20 mgm aeth.-test. daily by way of subcut. injection. Both dosages proved to be insufficient to maintain pregnancy.

A 2nd group of 4 rabbits received 40 mgm aeth.-test. subcut. daily and besides 10 mgm orally. Two of them were castrated on the 4th day and killed on the 10th or 12th. 4, resp. 8 nidated ova were found. The 2 other animals were castrated on the 12th day. One animal was killed 10 days later. Result: 1 living embryo and 5 on the way of resorption. The remaining animal was killed on the 17th day already: 9 normal fetusses were found. These experiments show that 40 mgm aeth.-test. subcut. and 10 mgm aeth.-test. orally, both daily administered, are able to make the nidation of the ovum possible in the castrate rabbit.

In one animal the authors could keep the fetus alive with this dosage during 5 days.

Our highest dosages were lower than the one with which COURRIER succeeded; the latter may probably be seen as a limit quantity.

This implies, that the relation between progesterone and aeth.-test. as found with the CLAUBERG test (1 : 4 : 8—10) does *not* pertain to the capacity to maintain pregnancy. COURRIER, when using aeth.-test. had to give the 15-fold of the active progesterone dose subcutaneously in order to keep the rabbit fetus alive. No details exist, concerning the required "oral" quantity.

e. Counteraction of oestrone in male mice: suppression of paradoxical effect.

It appeared from experiments by DE JONGH c.s. (15), that among the changes in the accessory reproductive organs of the spayed, adult mouse brought about under the influence of oestrone, one is specially counteracted by progesterone, namely the strong development of muco-reticular connective tissue around the ampulla ducti deferentis. With the aid of the qualitative and quantitative dates, resulting from these experiments, we investigated the influence of aeth.-test., orally administered, upon the

aforementioned paradoxical oestrone effect. Spayed adult male mice were injected with 1 γ oestrone, twice daily, for 3 weeks. One-half of the animals received besides daily aeth.-test., emulgated in gummi arabicum.

The rank growth of connective tissue around the ampulla after the injection of 2 γ oestrone daily for 3 weeks can be counteracted with 200 γ progesterone, subcutaneously given. The quantities of Aeth.-test., used by us and administered by way of a stomach tube, were 6—10 times as high, viz. 1.2—2 mgm daily.

The daily application of the stomach tube (in preliminary experiments even twice daily!) implies a severe trauma and caused a high mortality among our mice, so that, from 3 separate experiments, only 9 animals remained, that had been treated with a combination of oestrone and aeth.-test. and that could be compared with an equal number, treated with oestrone only. From these experiments it appeared, that the growth of the periampullar connective tissue could be slightly inhibited with 1.2 mgm aeth.-test. per os daily and distinctly with 2 mgm.

In view of this counteraction, the relation of the effective doses progesterone — aeth.-test. per os corresponds rather well with the one, found with the CLAUBERG-test.

f. Suppression of the oestrone-oestrus in female mice.

DE FREMERY, KOBER and TAUSK (16) reported, that in castrate, adult mice the oestrus, produced with 7 int. U oestrone (min. dosage for *oral* oestrus effect), was suppressed if 3 rabb. U oestronefree, non crystalline corpus luteum hormone ("progestine") was simultaneously injected (1 U: insufficient; 2 U: not experimented with).

In similar experiments we now investigated the influence of aeth.-test., orally given, on the oestrus, produced with a limit dose of oestrone, subcut. injected.

Castrate, female mice received 0.1 γ or 0.12 γ oestrone, divided over 3 injections (in 0.1 cc oil). One half of the animals received besides also in 3 injections 12 or 20 mgm aeth.-test. per os (stomach-tube). Vaginal smear every day, 3rd and 4th day twice daily. After a week the experiment was repeated, with interchange of the groups, in view of the given substances.

With 12 mgm aeth.-test. neither the oestrus effect of 0.1 γ nor that of 0.12 γ oestrone could be counteracted. In 3 experiments with groups of 5 animals, 8 had reached a positive oestrus index ("e, f. g.") at the moment of their maximal reaction [1]), under the influence of 0.1 γ oestrone, while the combination with 12 mgm aeth.-test. per os yielded 9 positive signs. In the experiment in which 0.12 γ oestrone had been used, the results even were resp. 4 and 11! The significance of these findings will be dealt with more closely in our next paper on the (oestrogenic) activity of aeth.-test.

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[1]) Details about this index: S. E. DE JONGH, E. LAQUEUR, P. DE FREMERY, Biochem. Zschr. 250, 1932.

With 20 mgm aeth.-test. orally a cross-test was done (2 series, each of 4 animals); with this quantity it proved to be possible to suppress the oestrus effect of 0.12 γ oestrone. The administration of solely 0.12 γ oestrone yielded 8 positive signs and only 2, when combined with 20 mgm aeth.-test *per os*. So 20 mgm aeth.-test. *per os* corresponds with 3 rabb. U progestine, (vid. the above research of DE FREMERY c.s.), as far as the inhibition of the artificial oestrus is concerned. When expressed in rabbit U (CLAUBERG test) aeth.-test. corresponds in our experiments with ca 2—3 R. U, so that a satisfying accordance has been found. However, besides a progesterone-like activity, aeth.-test. has testosterone-like properties and it is possible to counteract the oestrogenic effect of oestrone with testosterone also. For the discussion of the possibility, that the inhibition of the oestrone-oestrus with aeth.-test., rests upon a testosterone-like activity of aeth.-test., the reader is referred to our paper, dealing with this mechanism.

Summary.

The possibility of progesterone-like properties of aethinyl-testosterone (anhydro-oxyprogesterone, pregneninolone), subcutaneously or orally administered, has been studied. Mainly the following results were obtained:

1. In the CLAUBERG test (pregravid changes in the rabbit uterus) the effect of aeth.-test., subcutan. given, is ca $\frac{1}{4}$ that of progesterone and ca $\frac{1}{8}$—$\frac{1}{10}$ after oral administration.

2. It appeared to be impossible to provoke deciduomata in the immature rat and to maintain pregnancy in the rat and the rabbit after castration with aeth.-test. subcut. and aeth.-test. *per os,* resp. in dosages of 4 and 8—10 times the quantity, sufficient for progesterone.

3. Dates from literature (COURRIER) as well as our own experiments lead to the assumption that for the maintenance of pregnancy in the rabbit dosages of aeth.-test. are required, 15 times (subcut.) and 30—40 times (*per os*) the quantity sufficient for progesterone.

4. Aeth.-test., subcutaneously or orally administered, is capable of lowering the *sensitivity for oxytocine* of the rabbit uterus in the same quantitative relation, when compared with progesterone, as has been found with the CLAUBERG test.

5. Aeth.-test. *per os* is capable of inhibiting the effect of oestrone in castrated, male mice (paradoxical effect round ampulla ducti deferentis) and in castrated female mice (oestrus) in the same quantitative relation, when compared with progesterone, as has been found with the CLAUBERG test.

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Sérologie. — *Sur un nouveau phénomène observé dans la réaction de fixation.* (Deuxième mémoire.) Par Ong Sian Gwan. (Communicated by Prof. E. Gorter.) *)

(Communicated at the meeting of September 28, 1940.)

13. *Mécanisme du phénomène étudié.* On pourrait l'expliquer comme suite. Il existe dans le sang normal ou pathologique des sensibilisatrices antispermatozoïdes [1]) et tuberculeuses. Comme les anticorps sont des protéines et par conséquent des grosses molécules, il se peut q'une même molécule d'anticorps fixerait en même temps deux antigènes différents (fig. 1—C). Le complexe ainsi formé fixerait une quantité d'alexine plus

TABLEAU 7. Immunsérums.

Sérums			S	T	ST	ST — (S + T)
Sérum antitétanique, cheval R1			10 *	10	15 *	— 5
,,	,,	,, I.P.	10	10	10	0
,,	,,	,, 182	0	0	10	+ 10
,,	,,	,, R3	25	25	150	+100
,,	,,	, lapin 1	0(0)	0(5)	0(35)	0
,,	,,	,, 2	0	0	35	+ 35
,,	,,	,, 3	0(0)	0(0)	0(30)	0
,,	,,	,, 4	0(0)	0(0)	30(35)	+ 30
,,	,,	,, 6	10(0)	10(0)	15(35)	— 5
,,	antidiphtérique, cheval 182		20	10	35	+ 5
,,	,,	,, I.P.	0	0	15	+ 15
,,	antistreptococcique					
	polyvalent, cheval I.P.		0	10	15	+ 5
Sérum anticolibacillaire						
	polyvalent, lapin D7		20	20	35	— 5

Les chiffres entre parenthèses indiquent les résultats de réaction avant l'immunisation.
* Réactions effectuées avec l'antigène spermatozoïde dilué au 1/20.

élevée que celle obtenue par la somme des complexes anticorps-antigène spermatozoïde (fig. 1—A) et anticorps-antigène tuberculeux (fig. 1—B).

Dans le cas où l'une des deux ou toutes les deux réactions spermatozoïde et tuberculeuse sont négatives, on pourrait supposer que les sensibilisatrices

*) Premier mémoire, Proc. Ned. Akad. v. Wetensch., Amsterdam, **43**, 1133 (1940).

antispermatozoïde et tuberculeuse existent dans le sérum, si faibles soient
ils. Que le complexe anticorps-antigène tuberculeux-antigène spermatozoïde
(fig. 1—C) pourrait ainsi prendre naissance et qu'il serait capable de fixer
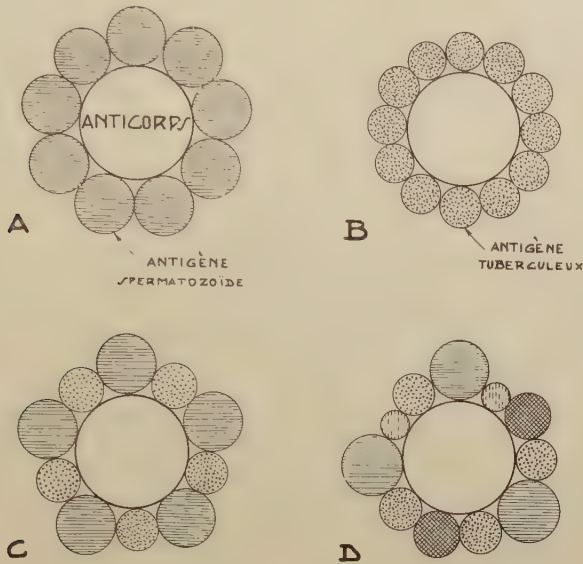

Fig. 1. A. Combinaison anticorps-antigène spermatozoïde. B. Combinaison
anticorps-antigène tuberculeux. C. Combinaison simultanée entre anticorps et
antigènes spermatozoïde et tuberculeux. D. Cimbinaison simultanée entre anti-
corps et plusieurs antigènes différents.

même fortement l'alexine. De cette façon pourrait être expliqué les résultats
de réactions des sérums du groupe I et du groupe II, A et B.

En généralisant l'hypothèse précédente, on pourrait supposer qu'un
anticorps serait également capable de combiner simultanément plusieurs
antigènes: a_1, a_2, ..., a_n (fig. 1—D) et qu'un tel complexe fixerait une
quantité d'alexine plus élevée que celle obtenue par la somme des
réactions réalisées avec les antigènes: a_1, a_2, ..., a_n séparément. Cette vue
se rapproche de celle de M. Kurt Meyer, qui a soustenu la conception
d'anticorps doué de plusieurs affinités [6]).

Une deuxième hypothèse également probable est la suivante: On pourrait
supposer que chaque molécule d'anticorps fixerait seulement l'antigène
correspondant (fig. 1—A ou B) et qu'un mélange de complexes différents
A et B fixerait une quantité d'alexine plus élevée que celle obtenue par la
somme des complexes A et B séparément. Cette deuxième hypothèse est
en désaccord avec les résultats obtenus sur le sérum syphilitique dilué dans
un sérum normal, contenant des sensibilisatrices antispermatozoïdes. Ici le
sérum normal ne contient pas de réagines syphilitiques et ne peut que
former le complexe anticorps-antigène spermatozoïde. Par contre le sérum

[6]) C. R. Soc. Biol. **122**, 26 (1936).

syphilitique dilué contient très peu ou pas de sensibilisatrices antispermatozoïdes et ne peut que former le complexe anticorps-antigène syphilitique.
Et pourtant le mélange de deux sérums en présence d'un mélange d'antigènes syphilitique et spermatozoïde ne dévie pas une quantité d'alexine
plus grande que celle obtenue par la somme des deux réactions séparément.

En supposant ces hypothèses vraies on devrait pouvoir diminuer et même
supprimer le phénomène, si l'on parviendrait à empêcher dans un mélange,
soit la fixation de l'un des deux antigènes par l'anticorps, soit la fixation
d'alexine par l'un des deux complexes A ou B. A cet effet, des réactions de
fixation ont été réalisées à des températures différentes: à 36° C dans
l'étuve, à la température de laboratoire variant de 15 à 22° C et à 0° C
(glace fondante). On sait que la fixation d'alexine diminue en même temps
avec la température et qu'à zéro degré la fixation est minime ou nulle.
Il est également admissible que la combinaison anticorps-antigène diminue
en même temps avec la température.

Pour effectuer la réaction à 0° C, il était nécessaire de prendre les précautions
suivantes: tous les réactifs et les tubes étaient plongés dans la glace fondante. Quand la
température du liquide restait constante on faisait le mélange et on laissait la réaction
s'effectuer pendant une heure dans la glace fondante. On ajoutait ensuite le système
hémolytique et on portait les tubes à l'étuve à 36° C.

Le tableau 8 montre les résultats de réactions de fixation avec 20 sérums
d'homme et un sérum antitétanique de cheval, effectuées à des températures
différentes. Ils montrent que, la différence $ST — (S + T)$ diminue en
même temps avec la température et qu'à 0° C elle est nulle ou négative
(excepté le sérum 81). Le phénomène est donc supprimé. La diminution de
réaction se manifeste surtout à 0° C dans la réaction de tuberculose; en
effet la plupart des réactions sont devenues négatives.

On en conclut que, si l'un des deux antigènes n'est plus ou faiblement
fixé par l'anticorps le phénomène ne se manifeste plus, ou bien que, si l'un
des deux complexes A ou B n'est plus capable de capturer l'alexine le
phénomène ne se montre pas non plus. Ces résultats ne montrent pas
laquelle des deux hypothèses est vraie; ils prouvent seulement qu'on peut
supprimer le phénomène, si l'on empêche l'une des deux réactions de se
réaliser dans le mélange d'antigènes spermatozoïde et tuberculeux.

14. *Le point de vue chimique du phénomène étudié.* On admet en
général que la réaction anticorps-antigène est un phénomène d'adsorption
(BORDET). Cependant pour expliquer la spécificité plusieurs auteurs
pensent qu'en dernier lieu il s'agirait d'une combinaison chimique. Pour
savoir si une combinaison chimique prend naissance dans le phénomène
considéré nous avons appliqué la détermination du pouvoir rotatoire. On
sait que, lorsque deux dissolutions de substances actives sont mêlées, sans
qu'il se produise une combinaison chimique, la déviation de polarisation est
égale à la somme des pouvoirs rotatoires des substances dissoutes chacune
dans le même volume de liquide. Au contraire, si une combinaison chimique

TABLEAU 8.

No des sérums	36° C				Température de laboratoire				0° C			
	S	T	ST	△	S	T	ST	△	S	T	ST	△
105	10	60	275	205	10	60	140	70	10	30	40	0
Sérum anti-tétanique R3	25	25	150	100	10	0	35	25	5	0	5	0
126	5	150	225	70	10	60	120	50	20	30	50	0
81	5	0	75	70	0	30	20	—10	5	0	10	5
77	5	5	55	45	5	0	5	0	5	0	0	— 5
114	10	45	90	35	10	40	70	20	10	30	40	0
35	20	30	85	35	30	15	60	15	15	0	15	0
36	10	15	60	35	10	10	20	0	10	0	10	0
110	0	30	55	25	0	30	50	20	0	15	15	0
63	0	5	30	25	0	0	15	15	0	0	0	0
73	5	0	30	25	0	0	10	10	0	0	0	0
82	0	5	30	25	10	20	30	0	5	0	5	0
103	0	0	25	25	5	10	20	5	0	0	0	0
111	0	0	20	20	0	0	20	20	10	0	10	0
104	0	30	45	15	0	20	55	35	10	0	10	0
127	10	40	60	10	10	20	35	5	10	0	5	— 5
106	10	10	30	10	0	5	15	10	5	0	5	0
107	10	10	30	10	0	0	15	15	0	0	0	0
72	5	15	30	10	0	15	20	5	0	0	0	0
52	5	10	25	10	10	10	25	5	0	0	0	0
124	10	15	30	5	10	20	40	10	25	10	25	—10

△ = différence ST—(S + T).
Les chiffres indiquent le nombre d'unités d'alexine déviées par 1 cc de sérum.

prend naissance, la déviation même l'accusera, et elle n'est plus égale à la somme des pouvoirs rotatoires de deux substances séparément.

Pour résoudre la question proposée nous avons déterminé le pouvoir rotatoire des mélanges suivants:

a. Sérum chauffé à étudier + mélange d'antigènes spermatozoïde et tuberculeux + alexine + eau physiologique Q. S. pour 4 cc.

b. Sérum chauffé à étudier + eau physiologique Q. S. pour 4 cc.

c. Mélange d'antigènes spermatozoïde et tuberculeux + eau physiologique Q. S. pour 4 cc.

d. Alexine + eau physiologique Q. S. pour 4 cc.

On a pris la précaution de mesurer exactement la même quantité de sérum, d'antigènes et d'alexine dans le tube réaction a et dans les tubes témoins b, c et d. Dans les expériences effectuées la quantité de réactifs utilisés était variable, de sorte qu'on pouvait mieux saisir le phénomène étudié. Les tubes étaient portés une à deux heures à l'étuve à 36° C et on procédait à la mesure de pouvoir rotatoire. Dans certains cas les tubes étaient gardés

pendant trois heures à l'étuve. Notons que le pouvoir rotatoire des mélanges ne variait pas après être conservés pendant 24 h. en glacière (+5° C).

Toutes les mesures furent effectuées au moyen d'un polaristrobomètre de WILD, dont le principe est un polariscope de SAVART placé entre deux nicols. En faisant 16 lectures sur chaque mélange, on obtient une moyenne avec un écart type variant de 0°,02 à 0°,07. Toutes les mesures ont été faites avec un même tube de 20 cm de longueur et de 0,5 cm de diamètre. Quatre centimètres cubes de liquide suffisent pour la mesure. La source de lumière était une lampe électrique à vapeur de sodium, fabriquée par Philips. ($\lambda \cong 5780$ Å).

Le tableau 9 représente les résultats de mesures avec 11 sérums, montrant une réaction de fixation très marquée.

TABLEAU 9. Sérums donnant des réactions de fixation très marquées.

No des sérums	Résultats des réactions de fixation			Réactions a_r en degrés	Témoins $a_1 + a_2 + a_3$ en degrés	Différence $a_r - (a_1 + a_2 + a_3)$ en degrés
	S	T	ST			
75	10	20	30	—0.44	—0.30	—0.14
77	5	5	55	—0.59	—0.66	+0.07
80	10	15	30	—0.08	—0.27	+0.19
106	10	10	30	—0.27	—0.16	—0.11
122	10	15	30	—0.77	—0.92	+0.15
123	15	15	30	—1.44	—1.51	+0.07
129	10	40	60	+0.06	—0.11	+0.17
130	10	15	30	—0.95	—0.82	—0.13
131	0	10	30	—0.95	—0.99	+0.04
133	10	10	30	—1.34	—1.39	+0.05
135	10	10	30	—2.57	—2.66	+0.09
Moyennes	9.1	15.0	35.0	—0.85	—0.89	+0.04

Si dans le mélange a une combinaison chimique prend naissance, on devrait obtenir une différence nette entre le pouvoir rotatoire du tube réaction a et la somme des pouvoirs rotatoires des tubes témoins b, c et d. La moyenne arithmétique des pouvoirs rotatoires des tubes réactions a est égale à $\bar{x} = -0°,85$ et celle de la somme des pouvoirs rotatoires des tubes témoins b, c et d est égale à $\bar{x}' = -0°,89$. Nous allons voir si la différence entre les deux moyennes $\bar{x}$ et $\bar{x}'$ est réelle.

On a: $\bar{x} - \bar{x}' = 0°,04$, $s = 0,751$, $t = 0,125$, $n = 20$.

Dans le tableau de t, on trouve pour $n = 20$, la probabilité $P \cong 0,9$ pour que, $t > 0,125$. La différence entre les deux moyennes n'est pas significative.

Puisque on a ici des observations parallèles, on pourrait aussi bien appliquer la deuxième méthode de la distribution de t. On trouve ainsi: $\bar{x} = +0,04$, $s = 0,368$, $t = 0,361$, $n = 10$.

La probabilité pour que, $t > 0,361$ serait entre 0,7 et 0,8. On trouve donc le même résultat que ci-dessus: la moyenne des différences entre les valeurs observées n'est pas réelle.

Par comparaison avec les sérums donnant des réactions de fixation fortement positives, nous avons examiné 13 sérums, dont les réactions sont négatives ou faiblement positives. Le sérum 138 avait donné un mélange hémorragique, on était obligé de le mesurer à la lumière rouge. Le tableau 10 montre les résultats obtenus. Le calcul donne: $\bar{x} - \bar{x}' = +0,04$, $s = 0,695$, $t = 0,147$, $n = 24$.

TABLEAU 10.

Sérums donnant des réactions de fixation négatives ou faiblement positives.

No des sérums	Résultats des réactions de fixation			Réactions a_r en degrés	Témoins $a_1 + a_2 + a_3$ en degrés	Différence $a_r - (a_1 + a_2 + a_3)$ en degrés
	S	T	ST			
100	0	0	0	—2.58	—2.27	—0.31
101	0	0	20	—2.47	—2.60	+0.13
102	0	0	10	—0.58	—0.49	—0.09
108	0	5	10	—0.88	—0.78	—0.10
109	0	0	10	—0.59	—0.68	+0.09
125	5	5	15	—0.72	—0.85	+0.13
128	0	0	15	—0.75	—0.89	+0.14
132	0	0	10	—1.17	—1.26	+0.09
134	0	0	15	—1.41	—1.78	+0.37
136	0	0	0	—2.18	—2.12	—0.06
137	0	5	20	—0.91	—1.23	+0.32
138	0	10	20	—1.38	—1.09	—0.29
140	0	0	0	—0.73	—0.85	+0.12
Moyennes	0	1.9	11.2	—1.26	—1.30	+0.04

Le tableau de t donne pour $n = 24$, la probabilité cherchée entre 0,8 et 0,9 pour que, $t > 0,147$. La différence entre les moyennes n'est pas significative.

La deuxième méthode de la distribution de t donne: $\bar{x} = +0,04$, $s = 0,647$, $t = 0,223$, $n = 12$.

La probabilité cherchée serait entre 0,8 et 0,9 pour que, $t > 0,223$. Les deux méthodes donnent le même résultat; la moyenne des différences n'est pas réelle.

En résumé, les sérums montrant une réaction de fixation fortement ou faiblement positive donnent le même résultat. Il n'y a aucune différence entre le pouvoir rotatoire du mélange considéré comme réaction et la somme des pouvoirs rotatoires des composants employés. Dans la réaction de fixation et, surtout dans le phénomène étudié, une combinaison chimique ne semble pas prendre naissance. Si nous avions trouvé une différence réelle entre les moyennes des pouvoirs rotatoires, on est sûr qu'il s'agit d'une combinaison chimique. Mais malgré les résultats négatifs obtenus, il est prudent de ne pas écarter la possibilité d'une intervention chimique. Dans cette

question si importante, touchant le mécanisme de la spécificité il est néces-
saire de vérifier les résultats par une autre méthode physique, la spectros-
copie.

15.   *Importance du phénomène observé.* a. En premier lieu on pourrait
probablement expliquer le mécanisme du phénomène de SANARELLI–
SHWARTZMAN. On sait que ce phénomène, comme celui de l'anaphylaxie,
consiste en deux étapes: une première injection préparante et une seconde
injection déchaînante. On pourrait expliquer le mécanisme de ce phénomène
comme suite: l'injection première et l'injection seconde donnent naissance
in vivo en présence d'anticorps normaux, des combinaisons anticorps-anti-
gènes du type C ou D. (fig. 1). Ces complexes devraient être pour l'animal
plus toxique que les complexes du type A ou B séparément, engendrés par
les filtrats microbiens et les anticorps normaux. Quand les injections
préparante et déchaînante sont de la même substance, il faudrait imaginer
l'intervention d'une deuxième substance, qui se trouverait dans l'organisme
de l'animal.

b.   L'allergie tuberculinique est probablement du même mécanisme. La
tuberculine pourrait former avec l'anticorps ou la cellule tuberculeuse avec
le concours d'une deuxième substance des combinaisons du type C ou D,
qui sont responsables à l'allergie tuberculinique.

Les échecs de l'épreuve de PRAUSNITZ–KÜSTNER et de la transfusion en
grosse quantité (8 à 10 cc) du sérum de cobaye tuberculeux et allergique
à un cobaye sain, que nous avons tenté en collaboration avec M. KOURILSKY
prouvent que, la deuxième substance ne se trouverait pas dans le sang du
cobaye tuberculeux [7]).

c.   L'observation suivante que nous avons fait autrefois pourrait égale-
ment ainsi expliqué. Elle consiste en ceci: un sérum anticolibacillaire poly-
valent agglutinant les souches employées à un titre variant de 1/300 à
1/32000, donne une agglutination maxima 1/32000 avec un mélange de
douze souches de B. coli employées à parties égales [8]). On pourrait penser
que les agglutinines colibacillaires sont fixées simultanément par les
différentes souches de B. coli, formant ainsi des combinaisons du type D.
Ces combinaisons provoqueraient une agglutination plus prononcée que les
complexes du type A ou B, constitué par chacune des souches employées
et les agglutinines colibacillaires.

d.   Enfin, le phénomène observé est en rélation avec ce qu'on est convenu
d'appeler la „synergie médicamenteuse". C'est-à-dire l'action d'un mélange
de deux médicaments est dans certains cas plus prononcée que la somme
des effets de deux médicaments séparément. En généralisant l'hypothèse
précédente, on pourrait expliquer le mécanisme ainsi: deux molécules
chimiques différentes, qui se combinent à une même cellule donneraient un

[7])   Revue de la tuberculose **3**, 350 (1937).
[8])   Zeitschr. f. Immunitätsf. **93**, 282 (1938).

effet plus grand que la somme des effets, résultant de la combinaison de la cellule avec les deux molécules chimiques séparément.

Ce travail était commencé en 1935 dans le service du professeur E. SERGENT, au laboratoire de recherches de la clinique propédeutique à l'hôpital Broussais-La Charité, Paris. Il fut ensuite poursuivi au laboratoire KAMERLINGH ONNES, grâce aux professeurs W. J. DE HAAS et W. H. KEESOM, directeurs du laboratoire, qui m'ont accueilli avec la plus grande bienveillance et dont je les remercie sincèrement. Je dois également remercier M. C. A. CROMMELIN, directeur-adjoint du laboratoire, qui a bien voulu mettre à ma disposition les appareils nécessaires.